MATHEMATICAL PHYSICS

M.Sc. PHYSICS

SEMESTER-I, PAPER-III

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FOREWORD

Since its establishment in 1976, Acharya Nagarjuna University has been forging ahead in the path of progress and dynamism, offering a variety of courses and research contributions. I am extremely happy that by gaining 'A+' grade from the NAAC in the year 2024, Acharya Nagarjuna University is offering educational opportunities at the UG, PG levels apart from research degrees to students from over 221 affiliated colleges spread over the two districts of Guntur and Prakasam.

The University has also started the Centre for Distance Education in 2003-04 with the aim of taking higher education to the door step of all the sectors of the society. The centre will be a great help to those who cannot join in colleges, those who cannot afford the exorbitant fees as regular students, and even to housewives desirous of pursuing higher studies. Acharya Nagarjuna University has started offering B.Sc., B.A., B.B.A., and B.Com courses at the Degree level and M.A., M.Com., M.Sc., M.B.A., and L.L.M., courses at the PG level from the academic year 2003-2004 onwards.

To facilitate easier understanding by students studying through the distance mode, these self-instruction materials have been prepared by eminent and experienced teachers. The lessons have been drafted with great care and expertise in the stipulated time by these teachers. Constructive ideas and scholarly suggestions are welcome from students and teachers involved respectively. Such ideas will be incorporated for the greater efficacy of this distance mode of education. For clarification of doubts and feedback, weekly classes and contact classes will be arranged at the UG and PG levels respectively.

It is my aim that students getting higher education through the Centre for Distance Education should improve their qualification, have better employment opportunities and in turn be part of country's progress. It is my fond desire that in the years to come, the Centre for Distance Education will go from strength to strength in the form of new courses and by catering to larger number of people. My congratulations to all the Directors, Academic Coordinators, Editors and Lesson-writers of the Centre who have helped in these endeavors.

Prof. K. Gangadhara Rao
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Acharya Nagarjuna University.

M.SC. PHYSICS
SYLLABUS
SEMESTER-I, PAPER-III
103PH24-MATHEMATICAL PHYSICS

Learning Objectives:

- ❖ Student should be able to understand basic theory of Complex Analysis, Special functions, Fourier series and integral transforms.
- ❖ To learn mathematical tools required to solve physical problem.
- ❖ To understand mathematical concepts related to physics
- ❖ To understand the relevance of higher mathematics and concepts of physics.

Unit-I

Beta & Gamma Functions - Definition, Relation between them- Properties.

Legendre's Differential Equation: The Power Series Solution-Legendre Functions of the first and second kind - Generating Function - Rodrigue's formula - Orthogonal Properties - Recurrence Relations - Physical applications.

Associated Legendre equation, Orthogonal properties of Associated Legendre's function.

Bessel's Differential Equation: Power series Solution -Bessel Functions of First and Second kind-Generating Function -Orthogonal Properties -Recurrence Relations- Physical applications.

Learning Outcomes:

- To learn about basic theory of polynomials
- To acquire knowledge about Legendre's, Associated Legendre's and Bessel equations.
- To learn the physical applications and properties in order to solve quantitative problems in the study of physics.

Unit-II

Hermite Differential Equation: Power Series Solution - Hermite Polynomials – Generating Function - Orthogonality - Recurrence relations - Rodrigues formula- Physical applications.

Laguerre Differential equations: The Power series Solution-Generating Function-Rodrigue's Formula- Recurrence Relations, Orthogonal Properties- Physical applications.

Learning Outcomes:

- To learn about basic theory of polynomials
- To acquire knowledge about Hermite Differential and Laguerre Differential Equation.
- To learn the physical applications and properties in order to solve quantitative problems in the study of physics.

Unit-III

Integral Transforms: Laplace Transforms - Definition - Properties - Derivative of Laplace Transform - Laplace Transform of a Derivative - Laplace Transform of Periodic Function - Evaluation of Laplace Transforms - Inverse Laplace transforms – Properties - Evaluation of Inverse Laplace transforms - Elementary Function Method - Partial Fraction Method - Solution of Ordinary Differential Equation by using Laplace Transformation Method - Fourier Series - Evaluation of Fourier Coefficients - Problems - Fourier Transforms - Infinite Fourier Transforms - Finite Fourier Transforms – Properties - Problems.

Learning Outcomes:

- This will enable students to apply integral transform to solve mathematical problems and used to understand the analysis of Fourier series.
- The students will be able to use Fourier transforms as an aid for analyzing different types of waves.

Unit-IV

Complex Variables: Function of Complex Number – Definition - Properties, Analytic Function - Cauchy - Riemann Conditions - Polar Form - Problems, Cauchy's Integral Theorem, Cauchy's Integral Formula - Problems, Taylor's Series-Laurent's Expansion - Problems, Calculus of Residues, Cauchy's Residue Theorem, Evaluation of Residues, Evaluation of Contour Integrals.

Learning Outcomes:

- To learn about complex algebra and Cauchy's integral theorems.
- To learn evaluation of contour integrals.

Unit-V

Tensor Analysis: Introduction - Contravariant, Covariant and Mixed Tensors - Rank of a Tensor - Symmetric and Anti-symmetric Tensors - Invariant Tensors, Addition and Multiplication of Tensors, Outer and Inner Products - Contraction of Tensors and Quotient Law.

Learning Outcomes:

- The students should be able to formulate and express a physical law in terms of tensors.
- To know how to simplify tensors by using coordinate transforms.
- To understand what extent tensors used to explain theory of relativity.

Course Outcomes:

After successfully completing the course, student will be able to:

- Understand the basic elements of complex analysis, including the important integral theorems.

- Understand the applications special functions that are used in quantum mechanics.
- Learned how to expand a function in a Fourier series and able to solve mathematical problems relevant to the physical sciences.

Text and Reference Books:

1) Mathematical Methods for Physics. By G.Arfken.
2) Laplace and Fourier Transforms - by Goyal and Gupta. Pragati Prakashan, Meerut.
3) Matrices and Tensors for Physicists by A.W. Joshi.
4) Mathematical Physics by B.D. Gupta, Vikas Publishing House, New Delhi.
5) Complex Variables, Schaum Series.
6) Vector and Tensor Analysis, Schaum Series.
7) Fundamentals of Mathematical Physics, 6th Edition by A.B.Gupta, Books and Allied, Kolkata.
8) Mathematical Physics - B.S. Rajput.
9) Mathematical Physics - Satya Prakash.

(103PH24)

M.Sc. DEGREE EXAMINATION, MODEL QUESTION PAPER
M.Sc. PHYSICS-FIRST SEMESTER
MATHEMATICAL PHYSICS

Time: Three hours **Maximum: 70 marks**

Answer ALL Questions

All Questions Carry Equal Marks

1 a) Show that $2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5 \ldots (2n-1)\sqrt{\pi}$?

b) Explain the power series solutions of Legendre's differential equations

OR

c) Briefly explain about the orthogonal properties of Associated Legendre's functions

d) Explain the Recurrence Relations of Bessel's differential equations

2 a) Explain the Hermite polynomials

b) Explain about the Rodrigue's formula of Laguerre differential equations

OR

c) Explain about the Laplace transforms of definition and properties

d) Explain about the properties and evaluation of Inverse Laplace transforms

3 a) Find the finite Fourier sine and cosine transforms of f(x) = 1 in (0, π).

b) If $z = re^{i\theta}$, show that the Cauchy – Riemann equations take the form

$$u_r = \frac{1}{r}v_\theta \text{ and } v_r = -\frac{1}{r}u_\theta$$

OR

c) Evaluate $\int_c \frac{z^2}{(z-5)}dz$, where "c" is the circle such that |z| = 2.

d) Expand Laurent's series $f(z) = \frac{1}{(z-1)(z-2)}$ $for\ 1 < |z| < 2$.

4 a) Evaluate real integral $\int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}dx$.

OR

b) Show that the law of transformation for a contravariant vector is transitive.

5 a) Define the rank of a tensor and provide examples of tensors with ranks 0, 1, and 2.

OR

b) Given a contravariant tensor A^{ij} and a covariant tensor B_k in a 3-dimensional coordinate system, where: $A^{ij} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$ and $B_k = \begin{pmatrix} k \\ k^2 \\ -k \end{pmatrix}$. Calculate the outer product $C^{ijk}=A^{ij}B_k$.

CONTENTS

LESSON-1

BETA AND GAMMA FUNCTIONS

1.0 AIM AND OBJECTIVE:

The primary aim of this lesson is to establish a comprehensive understanding of Beta and Gamma Functions, including their definitions, relationships, and essential properties. After completing this lesson, students should be able to define and apply the integral representations of both Beta and Gamma functions, derive and utilize the relationship between them, employ various properties to simplify and evaluate integrals and enhance their analytical and problem-solving skills through the application of these functions in diverse mathematical contexts.

STRUCTURE:

1.1 INTRODUCTION TO BETA AND GAMMA FUNCTIONS:

Beta and Gamma functions are powerful tools within mathematical analysis, acting as extensions of the factorial and offering elegant solutions to complex integrals. These special functions, defined through integral representations, reveal deep connections between seemingly disparate areas of mathematics, from probability and statistics to physics and number theory. Understanding their definitions, properties, and the inherent relationship between them equips students with a versatile toolkit for tackling a wide range of analytical problems.

1.2 BETA AND GAMMA FUNCTIONS:

DEFINITIONS:

Under the study of Definite Integrals, we come across two very important integrals known as Eulerian Integrals which are of the type

$$\int_0^\infty x^{m-1}(1-x)^{n-1}\,dx \text{ and } \int_0^\infty e^{-1}x^{n-1}\,dx$$

where the quantities m and n are supposed to be positive. These integrals are fundamental and hold an important place that they are widely applied in different branches of mathematical analysis like mechanics, physics etc.

The first Eulerian integral is generally known as Beta Function and defined as $\beta(m,n) = \int_0^\infty x^{m-1}(1-x)^{n-1}\,dx$ where m and n are positive.

The second Eulerian integral is known as Gamma Function and is defined as $\Gamma(n) = \int_0^\infty e^{-1}x^{n-1}\,dx$, where n is positive.

Note: Weierstrass (1815-1897) defined the Gamma function as

$$\frac{1}{\lceil(n)} = ne^{\gamma m}\prod_{m=1}^{\infty}\left[\left(1+\frac{n}{m}\right)e^{-n/m}\right]$$

where γ is known as Euler's or Mascheroni's constant and defined as

$$\gamma = \lim_{m\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\left(\frac{1}{m}-\log_e m\right)\right)$$

$$= \lim_{m\to\infty}(u_m - \log_e m)$$

with $u_m = \sum_{p=1}^{m}\frac{1}{p}$ and $\gamma = 0.5772157$ nearly.

1.3 RELATION BETWEEN BETA AND GAMMA FUNCTIONS:

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

From transformation of Gamma function, we have

$$\frac{\Gamma m}{\lambda^m} = \int_0^\infty e^{-\lambda x} x^{m-1}\, dx$$

i.e., $\Gamma m = \int_0^\infty \lambda^m e^{-\lambda x} x^{m-1}\, dx$

Multiplying both sides by $e^{-\lambda}\lambda^{n-1}$ and integrating w.r.t. λ within the limits 0 to ∞, we get

$$\Gamma m \int_0^\infty e^{-\lambda}.\lambda^{n-1}\, d\lambda = \int_0^\infty \left[\int_0^\infty e^{-\lambda(1+x)} \lambda^{m+n-1}\, d\lambda\right] x^{m-1}\, dx$$

or $\Gamma m\, \Gamma n = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}.x^{m-1}\, dx$ by equation from transformation of Gamma function

$$\Gamma m\, \Gamma n = \Gamma(m+n)\beta(m, n)$$

∴

$$\beta(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$$

1.4 FUNDAMENTAL PROPERTIES OF GAMMA FUNCTIONS:

$$\Gamma(n+1) = n\Gamma(n)$$

In order to prove this relation let us consider the integral

$$\int_0^\infty e^{-x} x^n\, dx = \Gamma(n+1)$$

Integrating it by parts taking e^{-x} as second function, we get

$$\int_0^\infty e^{-x} x^n\, dx = [-e^{-x} x^n]_0^\infty - \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= n \int_0^\infty e^{-x} x^{n-x}\, dx$$

(since $\frac{x^n}{e^x}$ vanishes for both the limits as $\text{Lim}_{x\to 0}\frac{x^n}{e^x} = 0$ and

$\text{Lim}_{x\to\infty}\frac{x^n}{e^x} = \text{Lim}_{x\to\infty}\frac{x^n}{1+x+\frac{x^2}{\lfloor 2}+\cdots} = 0$)

$$\therefore \qquad \Gamma(n+1) = n\Gamma(n) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

From (1) it is evident that if the value of $\Gamma(n)$ is known for n between two successive positive integers, then the value $\Gamma(n)$ for any positive value of n can be determined by the successive application of (1).

Now (1) can be written as

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If $-1 < n < 0$ then (2) gives $\Gamma(n)$, since n+1 is positive. As such the value of $\Gamma(n)$ may be determined if $-2 < n < -1$ since then $\Gamma(n+1)$ on the R.H.S. of (2) is known. Similarly, $\Gamma(n)$ may be determined when $-3 < n < -2$ and so on so forth.

Hence $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx = \frac{\Gamma(n+1)}{n}$ defines $\Gamma(n)$ completely for all values of n except $n = 0, -1, -2, -3, \ldots$

Now replacing n by $n-1$ in (1), we get

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

Similarly, $\Gamma(n-1) = (n-2)\Gamma(n-2)$ etc.

Hence (1) yields

$$\Gamma(n+1) = n(n-1)(n-2)\ldots 3.2.1\ \Gamma(1)$$

But by definition $\Gamma(1) = \int_0^\infty e^{-x}\,dx = [-e^{-x}]_0^\infty = 1$

$$\therefore \Gamma(n+1) = n(n-1)(n-2)\ldots 3.2.1 = \lfloor n. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Provided n is a positive integer.

Putting $n = 0$ in (3), we get

$$\Gamma(1) = \lfloor 0 = 1 \because \lfloor 0 = 1$$

$\therefore\ \Gamma(1) = 1$ ………………… (4)

Also, if we put $n = 0$ in (2), then we find

$\Gamma(0) = \frac{\Gamma(1)}{0} = \infty$ ………………… (5)

By repeated application of (2), it may be shown that the gamma function becomes infinite when n is zero or any negative integer, i.e.,

$\Gamma(-n) = \infty$ ………………… (6)

when $n = 0$ or a positive integer.

But the function has finite value for negative values of n which are not integer,

1.5 PROBLEMS:

1) Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{\Gamma m \Gamma n}{a^n(1+a)^m \Gamma(m+n)}$.

Let $I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx$

Put $\frac{x(1+a)}{(a+x)} = y$, so that $x = \frac{ay}{1+a-y}$

And $(1+a)\frac{a+x-x}{(a+x)^2}dx = dy$ i.e., $dx = \frac{(a+x)^2}{a(1+a)}dy$

As such $1 - x = 1 - \frac{ay}{1+a-y} = \frac{1+a-y-ay}{1+a-y} = \frac{(1+a)(1-y)}{1+a-y}$.

Also $a + x = a + \frac{ay}{1+a-y} = \frac{a+a^2-ay+ay}{1+a-y} = \frac{a(1+a)}{1+a-y}$.

And therefore, $dx = \frac{a^2(1+a)^2}{(1+a-y)^2 a(1+a)}dy = \frac{a(1+a)}{(1+a-y)^2}dy$.

Thus

$$I = \int_0^1 \frac{a^{m-1} y^{m-1}(1+a)^{n-1}(1-y)^{n-1}(1+a-y)^{m+n} a(1+a)}{(1+a-y)^{m-1}(1+a-y)^{n+1} a^{m+n}(1+a)^{m+n}(1+a-y)^2} dy$$

$$= \frac{1}{a^n(1+a)^m} \int_0^1 y^{m-1}(1-y)^{n-1} dy$$

or $I = \frac{1}{a^n(1+a)^m} \beta(m, n) = \frac{1}{a^n(1+a)^m} \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$.

2) Prove that $\int_0^{\pi/2} \sqrt{(tan\theta)} d\theta = \frac{\pi}{\sqrt{2}}$

L.H.S. = $\int_0^{\frac{\pi}{2}} \sqrt{(tan\theta)} d\theta$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{1/2} \times (\cos\theta)^{-1/2} d\theta$$

$$= \frac{2\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{2\Gamma 1} = \frac{\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{\sqrt{2}}.$$

1.6 SUMMARY:

This lesson delves into the Beta and Gamma functions, defining them through integral representations and exploring their fundamental properties. The Gamma function extends the factorial to complex numbers, while the Beta function facilitates the evaluation of specific integrals involving powers of variables. Crucially, the lesson establishes the relationship between these functions, allowing for the transformation and simplification of complex integrals. By mastering the definitions, properties, and interconnections of the Beta and Gamma functions, students gain valuable techniques for solving a wide range of analytical problems across various scientific and mathematical disciplines.

1.7 TECHNICAL TERMS:

Beta and Gamma Functions.

1.8 SELF-ASSESSMENT QUESTIONS:

1) Show that $2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5 \ldots (2n-1)\sqrt{\pi}$.

2) Show that $\int_0^1 \frac{d}{\sqrt{(1-x^n)}} = \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} \cdot \frac{\sqrt{\pi}}{n}$.

1.9 SUGGESTED BOOKS:

1) M.R. Spiegel ‘Complex variables’, McGraw – Hill Book co., 1964.

2) E. Kreyszig ‘Advanced engineering mathematics’, Wiley Eastern Pvt., Ltd., 1971.

3) B.D. Gupta ‘Mathematical Physics’, Vikas publishing House, Sahibabad, 1980.

Prof. R.V.S.S.N. Ravi Kumar

LESSON-2

LEGENDRE'S DIFFERENTIAL EQUATION

2.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Legendre's differential equation. The chapter began with understanding of The Power series Solutions, Legendre Functions of the first and second kind, Generating Functions, Rodrigue's formula, Orthogonal Properties, Recurrence Relations, Physical Application. After completing this chapter, the student will understand the complete idea about Legendre's Differential Equation.

STRUCTURE:

2.1 Introduction

2.2 The Power Series Solution

2.3 Legendre Functions of the First and Second Kind

2.4 Generating Function

2.5 Rodrigue's Formula

2.6 Orthogonal Properties

2.7 Recurrence Relations

2.8 Physical Applications

2.9 Summary

2.10 Key Terms

2.11 Self Assessments

2.12 Suggested Readings

2.1 INTRODUCTION:

Legendre's differential equation is a type of second-order ordinary differential equation (ODE) that arises in various areas of mathematical physics, particularly in problems with spherical symmetry. It is named after the French mathematician Adrien-Marie Legendre.

The standard form of **Legendre's differential equation** is:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Here:

- y=y(x) is the unknown function,
- n is a non-negative integer (often called the degree of the equation),
- x is the independent variable, typically with −1≤x≤1.

2.2 LEGENDRE POLYNOMIAL:

STATEMENT: The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

Where x is constant

This equation can be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0,$$

Where x is constant.

Solution of Legendre Equation (Power Series Solution):

(Power Series Method)

Let us assume the solution of eqn 1 in a series of descending powers of x as

$$y = \sum_{r=0}^{\infty} a_r\, x^{k-r} \text{-------} 2$$

derivationon both sides of 2

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k-r)\, x^{k-r-1}; \quad \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k-r)(k-r-1)\, x^{k-r-2}$$

Substitute the above values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in 1

We get

$$(1-x^2)\sum_{r=0}^{\infty} a_r (k-r)(k-r-1)\, x^{k-r-2} - 2x\sum_{r=0}^{\infty} a_r (k-r)\, x^{k-r-1}$$

$$+\, n(n+1)\sum_{r=0}^{\infty} a_r\, x^{t-r} = 0$$

$$\sum_{r=0}^{\infty} a_r \left[(k-r)(k-r-1)\, x^{k-r-2} + \{n(n+1) - (k-r)(k-r+1)\}x^{k-r}\right] = 0 \text{-----} 3$$

Eqn ③ being an identity, we can equate to zero the coefficients of various power of factors (x)

Let us first equate to zero the coefficient of highest power of x i.e, x^k (by putting r=0)

$$a_0 \{n(n+1) - k(k+1)\} = 0$$

$a_0 \neq 0$ [this is the 1st term of the series with which we can start the expansion]

$$a_0 \neq 0 \quad \therefore n(n+1) - k(k+1) = 0$$

$\therefore$ The indicial equation is $n(n+1)-k(k+1) = 0$

$$n^2 + n - k^2 + k = 0$$

$$(n^2 - k^2) + (n+k) = 0$$

$$(n-k) + (n+k+1) = 0$$

$$\therefore k = n \quad | \; k = -n-1 \text{---------} ④$$

Again equating to zero, the coefficient of next highest power of x i.e

x^{k-1}[by putting r=1]

$a_1\{ n(n+1) - (k-1)k\} = 0$

$\therefore$ $a_1 = 0$ and $n(n+1)-k(k-1) \neq 0$ [$\because$from ④]

Now equating to zero the coefficients of general term i.e., x^{k-r}

$$a_{r-2} (k-r+2)(k-r+1) + a_r \{ n(n+1) - (k-r)(k-r+1)\} = 0$$

$$a_r = \frac{-(k-r+2)(k-r+1)}{n(n+1)-(k-r)(k-r+1)} a_{r-2} \text{ ---------------} ⑤$$

CASE-I: When k =n the recurrence relation between the coefficients is given by

$$a_r = \frac{-(n-r+2)(n-r+1)}{(n^2+n-n^2-r^2+2nr-n+r)} a_{r-2} = \frac{-(n-r+2)(n-r+1)}{r(2n-r+1)} a_{r-2}$$

By putting r = 2,3,4,5……..

For r=2 $\quad a_2 = \frac{-(n)(n-1)}{2(2n-1)} a_0$

For r =3 $a_3 = \frac{-(n-1)(n-2)}{3(2n-2)} a_1 = 0$ [∵ $a_1 = 0$]

Similarly $a_5 = 0$, $a_7 = 0$, …….

∴ a 's having all 0dd suffixes are zero.

For r =4 $a_4 = \frac{-(n-2)(n-3)}{4(2n-3)} a_2$

$$a_4 = \frac{-(n-2)(n-3)}{4(2n-3)} \times \frac{-(n)(n-1)}{2(2n-1)} a_0 = \frac{n(n-1)(n-2)(n-3)}{2X4X(2n-1)(2n-3)} a_0$$

Similarly

$$a_6 = \frac{-n(n-1)(n-2)(n-3)(n-4)}{2X4X6X(2n-1)(2n-3)(2n-5)} a_0$$

By assuming solution from eq 2

$$y = \sum_{r=0}^{\infty} a_r x^{k-r}$$

Put k=n

$$y = \sum_{r=0}^{\infty} a_r x^{n-r}$$

$$y = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + a_4 x^{n-4} + \ldots\ldots$$

Substituting the values of a_2 , a_3 , a_4……., we get

$$y = a_0 \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2X4X(2n-1)(2n-3)} x^{n-4} + \ldots\ldots\right]$$

if $a_0 = \frac{1.3.5\ldots\ldots\ldots(2n-1)}{n!}$ In the above equation.

We get the solution of Legendre's differential equation and is represented by $P_n(x)$

$$P_n(x) = \frac{1.3.5\ldots\ldots\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2X4X(2n-1)(2n-3)} x^{n-4} + \ldots\ldots\right]$$

CASE-II: When k = -n-1 the recurrence relation between the coefficients i.e., eq 5

becomes

$$a_r = \frac{-(-n-r+1)(-n-r)}{n(n+1)-(-n-1-r)(-n-r)}\, a_{r-2} = \frac{-(n+r-1)(n+r)}{n^2+n-n^2-r^2-2nr-n-r}\, a_{r-2}$$

$$= \frac{(n+r-1)(n+r)}{r(2n+r+1)}\, a_{r-2}$$

By putting r = 2,3,4,…..

For r =2 $\quad a_2 = \frac{(n+1)(n+2)}{2(2n+3)}\, a_0$

For r = 3 $\quad a_3 = \frac{(n+2)(n+3)}{3(2n+4)}\, a_1 \quad$ [here $a_1 = 0$] -----4

$a_3 = 0 \quad [\because a_1 = 0]$

Similarly as having odd suffixes are zero.

$a_5 = a_7 = a_9 = \ldots\ldots\ldots = 0$ (each)

for r=4 $\quad a_4 = \frac{(n+3)(n+4)}{4(2n+5)}\, a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}\, a_0$

By assuming solution from eq 2

$y = \sum_{r=0}^{\infty} a_r\, x^{k-r}$

Put k = -n-1y $= \sum_{r=0}^{\infty} a_r\, x^{-n-1-r}$

Then $y = a_0 x^{-n-1} + a_1 x^{-n-2} + a_2 x^{-n-3} + \ldots\ldots$

Substituting the values of a_0 , a_1 , a_2 , ….. in the above expression, we get

$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \ldots\ldots\right]$$

if we get $a_0 = \frac{n!}{1.3.5\ldots\ldots(2n+1)}$ then above solution becomes second time solution of Legendre Polynomial and is denoted by $Q_n(x)$

$$\therefore\ Q_n(x) = \frac{n!}{1.3.5\ldots\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \ldots\ldots\right]$$

2.3 LEGENDRE FUNCTIONS OF THE FIRST AND SECOND KIND:

The **Legendre functions** of the first and second kind are solutions to **Legendre's differential equation**, which appears in various physics and engineering problems, especially in spherical coordinate systems.

Legendre's Differential Equation:

The equation is:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Where x is constant

Legendre Functions of the First Kind: $p_n(x)$

These are the solutions that are regular (finite) at x=±1 for integer values of n. For integer n, they are known as **Legendre polynomials**.

For example, the first few Legendre polynomials are:

- $p_0(x) = 1$
- $p_1(x) = x$
- $p_2(x) = \frac{1}{x}(3x^2 - 1)$
- $p_3(x) = \frac{1}{x}(5x^3 - 3x)$

These polynomials are widely used in physics for solving problems with spherical symmetry, like gravitational and electric potentials.

Legendre Functions of the Second Kind: $Q_n(x)$

These are the second linearly independent solutions to Legendre's equation and are typically singular (infinite) at x=±1. For integer n, they are denoted as$Q_n(x)$ and are less commonly used because of their singularity.

For example, for n=0and n=1, the functions are:

- $Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$
- $Q_1(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$-1

2.4 GENERATING FUNCTION OF LEGENDRE POLYNOMIAL:

Statement: $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

$$(1-2xh+h^2)^{-1/2} = [1-h(2x-h)]^{-1/2}$$

$$(1-x^n) = 1+nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots\ldots$$

$$(1-2xh+h^2)^{-1/2} = 1 + \frac{1}{2}h(2x-h) + \frac{\frac{1}{2}(\frac{1}{2}+1)}{2!}h^2(2x-h)^2 + \ldots\ldots + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\ldots\ldots\{\frac{1}{2}+(n-3)\}h^{n-2}(2x-h)^{n-2}}{(n-2)!}$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\ldots\ldots\{\frac{1}{2}+(n-2)\}h^{n-1}(2x-h)^{n-1}}{(n-1)!} + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\ldots\ldots\{\frac{1}{2}+(n-1)\}h^{n}(2x-h)^{n}}{(n)!}$$

$$= 1 + \frac{1}{2}h(2x-h) + \frac{1.3}{2^2\,2!}h^2(2x-h)^2 + \ldots\ldots\ldots + \frac{1.3.5\ldots\ldots\ldots(2n-5)}{2^{(n-2)}\,(n-2)!}h^{n-2}(2x-h)^{n-2} +$$

$$\frac{1.3.5\ldots\ldots\ldots(2n-3)}{2^{(n-1)}\,(n-1)!}h^{n-1}(2x-h)^{n-1} + \frac{1.3.5\ldots\ldots\ldots(2n-1)}{2^n\,n!}h^n(2x-h)^n$$

Now by collecting the coefficient of h^n from the above expansion, We have

$$(a-b)^n = {}^nC_0\, a^n b^0 - {}^nC_1\, a^{n-1} b^1 + \ldots\ldots\ldots$$

$$\frac{1.3.5\ldots\ldots\ldots(2n-1)}{2^n\,n!}2^n x^n - \frac{1.3.5\ldots\ldots\ldots(2n-3)}{2^{(n-1)}\,(n-1)!}\frac{(n-1)!}{(n-2)!}2^{n-2}x^{n-2} + \frac{1.3.5\ldots\ldots\ldots(2n-5)}{2^{(n-2)}\,(n-2)!}\frac{(n-2)!}{2!(n-4)!}2^{n-4}x^{n-4} - \ldots\ldots.$$

$$\frac{1.3.5\ldots\ldots\ldots(2n-1)}{n!}x^n - \frac{1.3.5\ldots\ldots\ldots(2n-3)}{2}\frac{(n-1)!}{(n-2)!}\frac{(2n-1)n(n-1)}{(2n-1)n(n-1)}x^{n-2} +$$

$$\frac{1.3.5\ldots\ldots\ldots(2n-5)}{2\,(n-4)!}\frac{(2n-3)(2n-1)n\,(n-1)(n-2)(n-3)}{(2n-3)(2n-1)n\,(n-1)(n-2)(n-3)}x^{n-4} - \ldots\ldots\ldots$$

$$\frac{1.3.5\ldots\ldots\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n\,(n-1)(n-2)(n-3)}{2.4.(2n-3)(2n-1)}x^{n-4} - \ldots..\right] = P_n(x)$$

$$\therefore (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

2.5 RODRIGUES FORMULA OF LEGENDRE POLYNOMIAL:

$$P_n(x) = \frac{1}{2^n\,n!}\frac{d^n(x^2-1)^n}{dx^n}$$

Proof:

Let $y = (x^2-1)^n$

$$\frac{dy}{dx} = n(x^2-1)^{n-1}(2x)$$

$(x^2-1)\frac{dy}{dx} = n(x^2-1)^{n-1}(x^2-1)(2x)$

$(x^2-1)\frac{dy}{dx} = 2nxy$

$D^n(AB) = {}^nC_0\, D^n(A).B + {}^nC_1\, D^{n-1}(A)\, D'(B) + {}^nC_n\, A\, D^n(B)$

${}^{(n+1)}C_0\frac{d^{n+2}y}{dx^{n+2}}(x^2-1) + {}^{(n+1)}C_1\frac{d^{n+1}y}{dx^{n+1}} \;.\, 2x + {}^{(n+1)}C_2\frac{d^ny}{dx^n} \;.\, 2 = 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2n(n+1)\frac{d^ny}{dx^n}$

$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2x\frac{d^{n+1}y}{dx^{n+1}} + \frac{n(n+1)}{2}\,2\,.\frac{d^ny}{dx^n} = 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2n(n+1)\frac{d^ny}{dx^n}$

$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^ny}{dx^n} = 0$

Put $Z = \frac{d^ny}{dx^n}$

$(1-x^2)\frac{d^2Z}{dx^2} - 2x\frac{dZ}{dx} + n(n+1)Z = 0$

This equation is in the form of Legendre differential equation. Hence the solution of this type of equation is

$Z = C\, P_n(x)$ ……….①

$\frac{d^ny}{dx^n} = C\, P_n(x)$

$[\frac{d^ny}{dx^n}]_{x=1} = C\, P_n(1)$

$[\frac{d^ny}{dx^n}]_{x=1} = C$ …….② $[\because\, P_n(1) = 1]$

Consider $y = (x^2-1)^n$

$y = (x+1)^n(x-1)^n$

Again differentiate by using Leibentz theorem upto n times

$\frac{d^ny}{dx^n} = {}^nC_0\,(x-1)^n\frac{d^n(x+1)^n}{dx^n} + {}^nC_1\, n(x-1)^{n-1}\frac{d^{n-1}(x+1)^n}{dx^{n-1}} + \ldots\ldots + {}^nC_n(x+1)^{nn}\frac{d^n(x-1)^n}{dx^n}$

Now

$$\frac{d^n y}{dx^n} = 2^n\, n!$$

$$\therefore \quad C = 2^n\, n! \quad [\text{from } ②]$$

From equation ①

$$Z = C\, P_n(x)$$

$$P_n(x) = \frac{Z}{C}$$

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n y}{dx^n}$$

$$\therefore \qquad P_n(x) = \frac{1}{2^n\, n!}\frac{d^n (x^2-1)^n}{dx^n}$$

2.6 ORTHOGONAL PROPERTY OF LEGENDRE POLYNOMIAL:

$$\int_{-1}^{+1} P_n(x) P_m(x)\, dx = \frac{2}{2^{n+1}}\delta_{mn}$$

Where δ_{mn} is called Kronecker delta function

$\delta_{mn} = 0$ when $m \neq n$

$\delta_{mn} = 1$ when $m = n$

Case-(1): $\boldsymbol{\int_{-1}^{+1} P_n(x) P_m(x)\, dx = 0}$

Proof: We know that Legendre differential equation

$$(1-x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

Let us assume that $P_n(x)$ & $P_m(x)$ are the two solutions of above equation

Then you may write

$\frac{d}{dx}[(1-x^2)\frac{dP_n(x)}{dx}]+n(n+1)P_n(x) = 0$……………….①

$\frac{d}{dx}[(1-x^2)\frac{dP_m(x)}{dx}]+m(m+1)P_m(x) = 0$ …………..②

Equation ① is multiplied with $P_m(x)$ & equation② with $P_n(x)$ then

$P_m(x)\frac{d}{dx}[(1-x^2)\frac{dP_n(x)}{dx}]+n(n+1)P_n(x)P_m(x) = 0$ …….③

$P_n(x)\ \frac{d}{dx}[(1-x^2)\frac{dP_m(x)}{dx}]+m(m+1)P_m(x)\ P_n(x)=0$ ……..④

Subtract equation ④ from equation ③ & then integrate w.r.t 'x' between the limits -1,+1

$\int_{-1}^{+1} Pm(x) \frac{d}{dx}[(1-x2)\frac{dP_n(x)}{dx}]dx - \int_{-1}^{+1} Pn(x)\ \frac{d}{dx}[(1-x2)\frac{dP_m(x)}{dx}]dx +$

$[n(n+1)-m(m+1)]\int_{-1}^{+1} P_n(x)P_m(x)\,dx = 0$

$[(1-x^2)\frac{dP_n(x)}{dx} P_m(x)]_{-1}^{+1} - \int_{-1}^{+1} \frac{dP_m(x)}{dx}[(1-x2)\frac{dP_n(x)}{dx}]dx - [[(1-x^2)\frac{dP_m(x)}{dx} P_n(x)]_{-1}^{+1}$

$+\int_{-1}^{+1} \frac{dP_n(x)}{dx}[(1-x2)\frac{dP_m(x)}{dx}]dx + [n(n+1)-m(m+1)]\int_{-1}^{+1} P_n(x)P_m(x)\,dx = 0$

$-[[(1-x^2)\frac{dP_m(x)}{dx} P_n(x)]_{-1}^{+1} - [(1-x^2)\frac{dP_n(x)}{dx} P_m(x)]_{-1}^{+1} + [n(n+1)-m(m+1)]\int_{-1}^{+1} P_n(x)P_m(x)\,dx = 0$

By applying limits we have

$[n(n+1)-m(m+1)]\int_{-1}^{+1} P_n(x)P_m(x)\,dx = 0,$

$\therefore \int_{-1}^{+1} P_n(x)P_m(x)\,dx = 0$, if $m \neq n$

Case-2: $\int_{-1}^{+1}[P_n(x)]^2\,dx = \frac{2}{2n+1}$ if m = n

We know that from the generating function of Legendre Polynomial

$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Squaring on both sides we get

$(1-2xh+h^2)^{-1} = \sum_{n=0}^{\infty} h^{2n}[P_n(x)]^2 + 2\sum_{m=0}^{\infty} h^{m+n} P_m(x)\,P_n(x)$

n=0

m≠n

Integrate the above equation w.r.t. 'x' between the limits -1 & +1.

$\int_{-1}^{+1} \frac{dx}{(1-2xh+h2)} = \sum_{n=0}^{\infty} h^{2n} [P_n(x)]^2 + 2 \sum_{n=0}^{\infty} h^{m+n} P_m(x) P_n(x)$

$= \frac{-1}{2h}$ [log($(1-2xh+h^2)]_{-1}^{+1}$ = $\sum_{n=0}^{\infty} h^{2n} [P_n(x)]^2$ + 0

[$\because \int_{-1}^{+1} P_n(x) P_m(x)\, dx = 0$, if m≠n]

$\frac{-1}{2h}[\log(1-h)^2 - \log(1+h)^2] = \sum_{n=0}^{\infty} h^{2n} [P_n(x)]^2$

$\frac{1}{h}[\log(1+h) - \log(1-h)] = \sum_{n=0}^{\infty} h^{2n} [P_n(x)]^2$

$\frac{1}{h} [(h-\frac{h^2}{2}+\frac{h^3}{3}-\frac{h^4}{4} + \text{-----}) - (-h+\frac{h^2}{2} - \frac{h^3}{3} + \frac{h^4}{4} - \ldots\ldots..)] = \sum_{n=0}^{\infty} h^{2n} [P_n(x)]^2$

$\Rightarrow \frac{1}{h} [2h + \frac{2h^3}{3} + \ldots\ldots\ldots.. + \frac{2h^{2n+1}}{2n+1} + \ldots\ldots\ldots..] = \sum_{n=0}^{\infty} h^{2n} [P_n(x)]^2$

$\Rightarrow 2[1+ \frac{h^2}{3} + \ldots\ldots\ldots\ldots. + \frac{h^{2n}}{2n+1} + \ldots\ldots\ldots..] = \sum_{n=0}^{\infty} h^{2n} [P_n(x)]^2$

Comparing co-efficients of h^{2n} on both sides, we have

$\int_{-1}^{+1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$ if m = n

PROBLEMS:

1) Show that $P_n(1) = 1$

$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Put x = 1

$(1-2h+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(1)$

$\sum_{n=0}^{\infty} h^n P_n(1) = ((1-h)^2)^{-1/2}$

$\sum_{n=0}^{\infty} h^n P_n(1)$ = $(1-h)^{-1}$

$\sum_{n=0}^{\infty} h^n P_n(1)$= 1 + h + h^2 + ………. + h^n +……..

Equating the coefficients of h^n on both sides

$$P_n(1) = 1$$

$$\therefore\ P_n(1) = 1$$

2) Show that $P_n(-x) = (-1)^n P_n(x)$ & $P_n(-1) = (-1)^n$

From the generating function of Legendre Polynomial

$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$---------①

Put x = -x

$(1+2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(-x)$--------②

Put h = -h in equation①

$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty}(-1)^n h^n P_n(x)$

Equating the coefficient h^n on both sides

$P_n(-x) = (-1)^n P_n(x)$

Put x = +1

$P_n(-1) = (-1)^n P_n(1)$

$\because P_n(1) = 1$

$P_n(-1) = (-1)^n$

2.7 RECURRENCE RELATION:

1. $(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$

2. $n\, P_n(x) = xP_n'(x) - P_{n-1}'(x)$

3. $(2n+1)\, P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

4. $(n+1)\, P_n(x) = P'_{n+1}(x) - xP_n'(x)$

5. $(1-x^2)P_n'(x) = n[\,P_{n-1}(x) - xP_n(x)\,]$

6. $(1-x^2)\, P_n'(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]$

7. $(2n+1)(x^2-1)\, P_n'(x) = n(n+1)[\,P_{n+1}(x) - P_{n-1}(x)]$

Proofs:

1) $(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$

From the generating function of Legendre Polynomial we have

$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n\, P_n(x)$

Differentiate w.r.t. 'h' on both sides

$\frac{-1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$

$(x-h)\, \frac{(1-2xh+h2)^{\frac{-1}{2}}}{(1-2xh+h2)} = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$

$(x-h)\sum_{n=0}^{\infty} h^n\, P_n(x) = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$

$X\sum_{n=0}^{\infty} h^n\, P_n(x) - h\, \sum_{n=0}^{\infty} h^n\, P_n(x) = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x) - 2xh\, \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$

$+ h^2\sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$

Comparing the coefficients of h^n on both sides

$xP_n(x) - P_{n-1}(x) = (n+1)\, P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$

$2xnP_n(x) + xP_n(x) - P_{n-1}(x) - nP_{n-1}(x) + P_{n-1}(x) - (n+1)\, P_{n+1}(x) = 0$

$\therefore (2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$

2) $n\, P_n(x) = xP_n'(x) - P_{n-1}'(x)$

From the generating function of Legendre Polynomial we have

$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$---------1

Differentiate w.r.t. 'h' On both sides

$\frac{-1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$

$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$-------2

$\frac{-1}{2}(1-2xh+h^2)^{-3/2} (-2h) = \sum_{n=0}^{\infty} h^n P_n'(x)$

$h (1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x)$---------3

dividing equation 2 by 3

$\frac{(x-h)}{h} = \frac{\sum_{n=0}^{\infty} nh^{n-1} P_n(x)}{\sum_{n=0}^{\infty} h^n P_n'(x)}$

$(x-h) \sum_{n=0}^{\infty} h^n P_n'(x) = h \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$

$x \sum_{n=0}^{\infty} h^n P_n'(x) - h\sum_{n=0}^{\infty} h^n P_n'(x) = h \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$

equating the co-efficient of h^n on both sides

$x P_n'(x) - P_{n-1}'(x) = n P_n(x)$

$\therefore n P_n(x) = xP_n'(x) - P_{n-1}'(x)$

3) $(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

From the first recurrence relation we have

$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$

Differentiate above equation w.r.t 'x' on both sides

$(2n+1) x P'_n(x) + (2n+1) P_n(x) = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$

$(2n+1)[x P'_n(x) + P_n(x)] = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$

Consider recurrence relation 2

$n P_n(x) = xP_n'(x) - P_{n-1}'(x)$

$x P'_n(x) = n P_n(x) + P_{n-1}'(x)$

Subtract value $x P'_n(x)$ from equation 2 in equation 1

$(2n+1)[n P_n(x) + P_{n-1}'(x) + P_n(x)] = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$

$(2n+1)(n+1) P_n(x)+ (2n+1) P'_{n-1}(x) = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$

$(2n+1)(n+1) P_n(x) +(2n+1-n)P'_{n-1}(x) = (n+1) P'_{n+1}(x)$

$(2n+1)(n+1) P_n(x) + (n+1)P'_{n-1}(x) = (n+1) P'_{n+1}(x)$

$(2n+1)P_n(x) + P'_{n-1}(x) = P'_{n+1}(x)$

$\therefore (2n+1) P_n(x) = P'_{n+1}(x)-P'_{n-1} (x)$

4) $(n+1) P_n(x) = P'_{n+1}(x)- xP_n'(x)$

From '3' recurrence relation we have

$(2n+1) P_n(x) = P'_{n+1}(x)-P'_{n-1} (x)$……….1

From '2' recurrence relation we have

$n P_n(x) = xP_n'(x) - P_{n-1}'(x)$………..2

Subtracting 2 from 1

1 - 2

$(2n+1) P_n(x) - n P_n(x) = P'_{n+1}(x)-P'_{n-1} (x)-xP_n'(x) + P_{n-1}'(x)$

$(2n+1-n) P_n(x) = P'_{n+1}(x)-xP_n'(x)$

$\therefore(n+1) P_n(x) = P'_{n+1}(x)- xP_n'(x)$

5) $(1-x^2) P_n'(x) = n[P_{n-1} (x) - xP_n(x)]$

From the fourth recurrence relation

$(n+1) P_n(x) = P'_{n+1}(x)- xP_n'(x)$

Replacing 'n' by n-1

$(n-1+1)P_{n-1}(x) = P'_{n-1+1}(x) - xP'_{n-1}(x)$

$nP_{n-1}(x) \quad = P_n'(x) - xP'_{n-1}(x)$ ……….1

From the '2' recurrence relation

$n P_n(x) = xP_n'(x) - P_{n-1}'(x)$

multiplying above equation with x

$nx P_n(x) = x^2P_n'(x) - xP'_{n-1}(x)$ ……… 2

Subtract 1 - 2

$n P_n(x) - nx P_n(x) = P_n'(x) - x^2P_n'(x) - xP'_{n-1}(x) + xP'_{n-1}(x)$

$\Rightarrow n [P_{n-1}(x) - x P_n(x)] = (1-x^2)P_n'(x)$

$\therefore (1-x^2)P_n'(x) = n [P_{n-1}(x) - x P_n(x)]$

6) $(1-x^2) P_n'(x) = (n+1) [xP_n(x) - P_{n+1}(x)]$

From '1' recurrence relation

$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$

$(n+n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$

$(n+1) [x P_n(x) - P_{n+1}(x)] = n [P_{n-1}(x) - x P_n(x)]$

From 5th recurrence relation

$(1-x^2)P_n'(x) = n [P_{n-1}(x) - x P_n(x)]$

From equation 1&2

$(1-x^2)P_n'(x) = (n+1) [xP_n(x) - P_{n+1}(x)]$

7) $(2n+1)(x^2-1) P_n'(x) = n(n+1)[P_{n+1}(x) - P_{n-1}(x)]$

From 5th recurrence relation

$(1-x^2)P_n'(x) = n\,[\,P_{n-1}(x) \quad - x\,P_n(x)\,]$

$\frac{(1-x2)}{n}P_n'(x) = P_{n-1}(x) \quad - x\,P_n(x)$

$x\,P_n(x) = P_{n-1}(x) - \frac{(1-x2)}{n}P_n'(x)$ ………①

From 6th recurrence relation

$(1-x^2)P_n'(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]$………②

Substitute the value of x $P_n(x)$ from ① in ②

$(1-x^2)P_n'(x) = (n+1)\,[P_{n-1}(x) - \frac{(1-x2)}{n}P_n'(x) - P_{n+1}(x)]$

$(\frac{n+1}{n} + 1)(1-x^2)P_n'(x) = (n+1)\,[P_{n-1}(x) - P_{n+1}(x)]$

$(2n+1)(1-x^2)P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$

Write the values of $P_1(x)$, $P_2(x)$, $P_3(x)$ & $P_4(x)$

$P_1(x)$

From Rodrigues formula of Legendre Polynomial

$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n(x^2-1)^n}{dx^n}$

For n = 1 $P_1(x) = \frac{1}{2\ .1!}\frac{d^1(x^2-1)^1}{dx^1}$

⇨$P_1(x) = \frac{1}{2}\frac{d(x^2-1)}{dx}$

⇨$P_1(x) = \frac{1}{2}\,.2x$

⇨ $P_1(x) = x$

$P_2(x)$

From Rodrigues formula of Legendre Polynomial

$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n(x^2-1)^n}{dx^n}$

For n=2

$P_2(x) = \frac{1}{2^2 2!}\frac{d^n(x^2-1)^2}{dx^2}$

$P_2(x)= \frac{1}{4\ x\ 2}\frac{d^n(x^2-1)^2}{dx^2}$

$P_2(x) = \frac{1}{8}\frac{d[2.2x.(x^2-1)]}{dx}$

$P_2(x) = \frac{1}{2}[\ (x^2-1) + x.2x]$

$P_2(x) = \frac{1}{2}[\ 2x^2+x^2-1]$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x)$

$P_n(x) = \frac{1}{2^n\ n!}\frac{d^n(x^2-1)^n}{dx^n}$

For n=3

$P_3(x) = \frac{1}{8x6}\frac{d^3(x^2-1)^3}{dx^3}$

$P_3(x) = \frac{1}{48}\frac{d}{dx}[\frac{d^2(x^2-1)^3}{dx^2}]$

$P_3(x) = \frac{1}{48}\frac{d^2[3(x^2-1)^2.2x]}{dx^2}$

$= \frac{1}{48}\frac{d^2[6x(x^2-1)^2]}{dx^2}$

$= \frac{1}{8}\frac{d}{dx}[\frac{d}{dx}[x.(x^2-1)^2\ ]$

$= \frac{1}{8}\frac{d}{dx}[\ (x^2-1)^2 + 2\ x(x^2-1).2x]$

$= \frac{1}{8}\frac{d}{dx}[\ (x^2-1)^2 + 4x^2(x^2-1)]$

$= \frac{1}{8}[2x.2(x^2-1)+ 4.2x\ (x^2-1) + 4x^2.2x]$

$= \frac{1}{8}[4x^3\text{-}4x+8x^3\text{-}8x+8x^3]$

$P_3(x) = \frac{1}{2}[5x^3\text{-}3x]$

$P_4(x) = \frac{1}{16x24}\frac{d^4(x^2-1)^4}{dx^4}$

$= \frac{1}{16x24}\frac{d^3 4(x^2-1)^3\,2x}{dx^3}$

$= \frac{1}{2x24}\frac{d^3 x(x^2-1)^3}{dx^3}$

$= \frac{1}{2x24}\frac{d^2[x.3(x^2-1)^2.2x+(x^2-1)^3]}{dx^2}$

$= \frac{1}{2x24}\frac{d^2[6x^2.3(x^2-1)^2+(x^2-1)^3]}{dx^2}$

$= \frac{1}{2x24}[6.\frac{d}{dx}[2x.(x^2-1)^2+x^2.2(x^2-1).2x + 3\,(x^2-1)^2.2x]$

$= \frac{1}{8}[\frac{d}{dx}[2x.(x^2-1)^2+4x^3.2(x^2-1) + x.(x^2-1)^2]$

$= \frac{1}{8}[\,2.(x^2-1)^2+2x.2(x^2-1).2x+4.3x^2(x^2-1)+4x^3.2x+(x^2-1)^2+x.2(x^2-1).2x]$

$=\frac{1}{8}[2.(x^4+1\text{-}2x^2)+8x^4\text{-}8x^2+12x^4\text{-}12x^2+ 8x^4+x^4+1\text{-}2x^2+4x^4\text{-}4x^2]$

$= \frac{1}{8}[2x^4\text{-}4x^2+2+8x^4\text{-}8x^2+12x^4\text{-}12x^2+8x^4+5x^4\text{-}6x^2+1]$

$P_4(x) = \frac{1}{8}[35x^4\text{-}30x^2+3]$

$P_5(x) = \frac{1}{8}[63\,x^5\text{-}70x^3+15x]$

2.8 PHYSICAL APPLICATIONS:

Legendre's differential equation appears in various **physical applications**, especially in systems with **spherical symmetry**. Here are some key physical applications where this equation and its solutions play a crucial role:

Electrostatics:

In electrostatics, Legendre's equation arises when solving **Laplace's equation** in spherical coordinates for systems with azimuthal symmetry. For instance, when determining the electric potential $\Phi(r,\theta)$ in the absence of free charge:

$\nabla^2\ \Phi$=0

The solution often separates into radial and angular components. The angular part leads to Legendre's equation:

$$\frac{d}{d\theta}sin\theta\frac{d\Phi}{d\theta} + n(n+1)sin\theta\Phi$$

Here, the solutions $p_n(cos\theta)$ describe the angular dependence of the potential.

Gravitational Potential:

Similar to electrostatics, the gravitational potential around a spherical mass distribution can be described by Laplace's equation. The angular part again leads to Legendre functions, describing the potential's angular variation due to non-uniform mass distributions.

Heat Conduction in Spherical Objects:

In steady-state heat conduction problems for spherical objects, Legendre's equation arises when the temperature distribution depends on the polar angle θ\thetaθ, such as in spherical shells or spheres with azimuthal symmetry.

Acoustics and Vibrations:

In acoustics, when studying sound waves in spherical cavities or around spherical objects, the wave equation in spherical coordinates often leads to Legendre's differential equation for the angular component.

Antenna Theory:

Legendre functions model the radiation patterns of antennas, especially when the antenna has spherical symmetry or when analyzing the far-field radiation in terms of angular distribution.

Geophysics:

Legendre polynomials are used to describe Earth's gravitational and magnetic fields. Models like the **geoid** (Earth’s equipotential surface) use spherical harmonics, where Legendre functions play a central role.

Optics:

In optical systems with spherical mirrors or lenses, Legendre functions describe the angular variation of light intensity, especially when considering diffraction effects.

Celestial Mechanics:

When studying planetary orbits and gravitational interactions in celestial mechanics, Legendre polynomials help expand the gravitational potential due to non-spherical mass distributions (e.g., Earth's oblateness).

Magnetostatics:

Legendre functions describe magnetic fields around spherical conductors or in systems where the magnetic potential has spherical symmetry.

These examples highlight how Legendre's equation and its solutions are integral to modeling and solving complex physical phenomena in systems exhibiting spherical symmetry.

2.9 SUMMARY:

Legendre's differential equation is a second-order linear ordinary differential equation commonly encountered in physics and engineering, especially in problems exhibiting spherical symmetry. The equation is:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

Where x is constant.

2.10 KEY TERMS:

1) **Legendre Polynomials**: When n is a non-negative integer, the solutions to this equation are the **Legendre polynomials**, denoted by $p_1(x)$.. These are a set of orthogonal polynomials over the interval x∈[−1,1].

2) **General Solution**: For integer n, the general solution is:

   $y(x) = Ap_1(x) + BQ_1(x)$..

   Where:

   - $p_1(x)$ is the Legendre polynomial of degree lll,
   - $Q_1(x)$is the Legendre function of the second kind, which often has singularities at x=±1
   - A and B are constants determined by boundary conditions.

3) **Rodrigues' Formula**: Legendre polynomials can be explicitly calculated using Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n(x^2-1)^n}{dx^n}$$

4) **Orthogonality**: The Legendre polynomials satisfy the orthogonality relation:

$$\int_{-1}^{+1} P_n(x)P_m(x)dx = \frac{2}{2^{n+1}}\delta_{mn}$$

2.11 SELF ASSESSMENT QUESTIONS:

1) Explain about the power series solutions of Legendre’s equation?
2) Briefly explain about the Generating function of Legendre’s equation?
3) Explain about the Orthogonal properties of Legendre’s equation?

2.12 SUGGESTED READINGS:

1) **“Mathematical Methods for Physicists”** by George B. Arfken, Hans J. Weber, and Frank E. Harris
    - This book offers a thorough treatment of special functions, including Legendre polynomials, and covers their derivation and applications in physics.
2) **“Advanced Engineering Mathematics”** by Erwin Kreyszig
    - Kreyszig's book provides detailed explanations of Legendre’s equation in the context of solving boundary value problems and includes practical engineering applications.
3) **“Mathematical Methods in the Physical Sciences”** by Mary L. Boas
    - Boas provides an accessible introduction to special functions, including Legendre polynomials, with applications in physics and engineering.
4) **“Applied Partial Differential Equations”** by Richard Haberman
    - This book discusses Legendre's equation in the context of solving partial differential equations, particularly Laplace’s and Poisson’s equations in spherical coordinates.

Prof. R.V.S.S.N. Ravi Kumar

LESSON-3

ASSOCIATED LEGENDRE EQUATION

3.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Associated Legendre equation. The chapter began with understanding of Associated Legendre equation, Orthogonal Properties of Associated Legendre’s function. After completing this chapter, the student will understand the complete idea about Associated Legendre Equation.

STRUCTURE:

3.1 Introduction

3.2 Associated Legendre Equation

3.3 Associated Legendre Function

3.4 Orthogonal Properties of Associated Legendre’s Function

3.5 Summary

3.6 Key Terms

3.7 Self Assessment Questions

3.8 Suggested Readings

3.1 INTRODUCTION:

The **associated Legendre equation** is a second-order linear differential equation that arises frequently in the context of solving physical problems, particularly in **spherical coordinates**. It is often encountered in problems involving angular parts of solutions to Laplace's equation, such as in the case of **spherical harmonics.**

3.2 ASSOCIATED LEGENDRE EQUATION:

The differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(n(n+1)-\frac{m^2}{1-x^2})y=0$$

If ‘v’ is the solution of legendre equation then

$(1-x^2)^{\frac{m}{2}}\frac{d^my}{dx^m}$ is the solution of associated legendre equation

Proof:-

Since 'p' is the solution of Legendre equation

We have

$(1-x^2)\frac{d^2v}{dx^2}$-2x$\frac{dv}{dx}$+n(n+1)v=0

Derivative above equation with respect to 'x' upto to 'm' times using Leibentz theorem,we have

$(1-x^2)\ \frac{d^{m+2}v}{dx^{m+2}}$ - 2mx$\frac{d^{m+1}v}{dx^{m+1}}$-$\frac{2m(m-1)}{2!}\ \frac{d^m v}{dx^m}$-2{x$\frac{d^{m+1}v}{dx^{m+1}}$+m$\frac{d^m v}{dx^m}$} + n(n+1)$\frac{d^m v}{dx^m}$=0

$(1-x^2)\ \frac{d^{m+2}v}{dx^{m+2}}$ - 2(m+1) x $\frac{d^{m+1}v}{dx^{m+1}}$ - 2m [$\frac{(m-1+1)}{2!}$]$\frac{d^m v}{dx^m}$ + {n(n+1)-m(m+1)}$\frac{d^m v}{dx^m}$=0

Let us take v_1 for $\frac{d^m v}{dx^m}$

$(1-x^2)\ \frac{d^2v_1}{dx^2}$ - 2(m+1) x$\frac{dv_1}{dx}$ + {n(n+1)-m(m+1)}v_1=0 ………..1

Now let

Z=$(1-x^2)^{\frac{m}{2}}\frac{d^m v}{dx^m}$

Z=$(1-x^2)^{\frac{m}{2}}v_1$

v_1=z$(1-x^2)^{\frac{-m}{2}}$

Derivative with respect to 'x'

$\frac{dv_1}{dx}$=z($\frac{-m}{2}$)(-2x)$(1-x^2)^{-\frac{m}{2}-1}\ \frac{dz}{dx}$

$\frac{dv_1}{dx} = (1-x^2)^{-\frac{m}{2}}\frac{dz}{dx}$+mxz$(1-x^2)^{-\frac{m}{2}-1}$

$\frac{d^2v_1}{dx^2} = \frac{+m}{2}(1-x^2)^{-\frac{m}{2}-1}$(2x)$\frac{dz}{dx}$+$(1-x^2)^{-\frac{m}{2}-1}\ \frac{d^2z}{dx^2}$+mx$(1-x^2)^{-\frac{m}{2}-1}\ \frac{dz}{dx}$+mz

$(1-x^2)^{-\frac{m}{2}-1}\frac{dz}{dx}+mz(1-x^2)^{-\frac{m}{2}-1}$+mxz($\frac{-m}{z}$-1) $(1-x^2)^{-\frac{m}{2}-2}$(-2x)

$\frac{d^2v_1}{dx^2}$=mx$(1-x^2)^{-\frac{m}{2}}\frac{dz}{dx}$ +$(1-x^2)^{-\frac{m}{2}}\frac{d^2z}{dx^2}$ +mx$(1-x^2)^{-\frac{m}{2}-1}\frac{dz}{dx}$+mz$(1-x^2)^{-\frac{m}{2}-1}$-2m$x^2$z($\frac{-m}{2}$-1) $(1-x^2)^{-\frac{m}{2}-2}$

$\frac{d^2v_1}{dx^2}$ = $(1-x^2)^{-\frac{m}{2}}\frac{d^2z}{dx^2}$+2mx$(1-x^2)^{-\frac{m}{2}}\frac{dz}{dx}$ +mz$(1-x^2)^{-\frac{m}{2}-1}$+m(m+2)x^2z $(1-x^2)^{-\frac{m}{2}-2}$

Multiplying the above equation with$(1-x^2)$

$(1-x^2)\frac{d^2v_1}{dx^2}$ = $(1-x^2)^{\frac{-m}{2}+1}\frac{d^2z}{dx^2}$+2mx$(1-x^2)^{-\frac{m}{2}}\frac{dz}{dx}$ +mz$(1-x^2)^{-\frac{m}{2}}$+m(m+2)x^2z $(1-x^2)^{-\frac{m}{2}-1}$

$(1-x^2)^{\frac{-m}{2}}$[$(1-x^2)\frac{d^2z}{dx^2}$+2mx$\frac{dz}{dx}$ +mz+m(m+2)x^2z $(1-x^2)$]

Sub above values in equation 1 then

$(1-x^2)^{\frac{-m}{2}}$[$(1-x^2)\frac{d^2z}{dx^2}$+2mx$\frac{dz}{dx}$+mz+$\frac{m(m+2)x^2z}{(1-x^2)}$]-2(m+1)x[$(1-x^2)^{-\frac{m}{2}}\frac{dz}{dx}$

+mxz$(1-x^2)^{-\frac{m}{2}-1}$]+{n(n+1)-m(m+1)} $(1-x^2)^{\frac{-m}{2}}$z=0

$(1-x^2)^{\frac{-m}{2}}$[$(1-x^2)\frac{d^2z}{dx^2}$+2mx$\frac{dz}{dx}$+mz+$\frac{m(m+2)x^2z}{(1-x^2)}$]-2(m+1) x$\frac{dz}{dx}$ - $\frac{2x^2m(m+1)z}{(1-x^2)}$

+[n(n+1)-m(m+1)z]=0

$(1-x^2)\frac{d^2z}{dx^2}$-2x$\frac{dz}{dx}$+mz+$\frac{mx^2z[m+2-2(m+1)]}{(1-x^2)}$ +n(n+1)z-m(m+1)z=0

$(1-x^2)\frac{d^2z}{dx^2}$-2x$\frac{dz}{dx}\frac{m^2x^2z}{(1-x^2)}$+n(n+1)z-$m^2$z=0

$(1-x^2)\frac{d^2z}{dx^2}$-2x$\frac{dz}{dx}$ - m^2z ($\frac{x^2}{(1-x^2)}+1$)+n(n+1)z=0

$(1-x^2)\frac{d^2z}{dx^2}$-2x$\frac{dz}{dx}$ - $\frac{m^2z}{1-x^2}$ +n(n+1)z=0

$(1-x^2)\frac{d^2z}{dx^2}$-2x$\frac{dz}{dx}$ - +n((n+1)-$\frac{m^2}{1-x^2}$)z=0

Hence z=$(1-x^2)^{+\frac{m}{2}}\frac{d^m v}{dx^m}$ is the solution of associated legendre solution

Note:

Put x=cosθ the equation can be written as $\frac{1}{sin\theta}\frac{\partial}{\partial\theta}$(sinθ$\frac{\partial y}{\partial\theta}$)$\frac{1}{sin^2\theta}\frac{\partial^2 y}{\partial^2\theta^2}$

+[n(n+1)-$\frac{m^2}{1-cos^2\theta}$]y=0

Y=cp^m(cosθ)

3.3 ASSOCIATED LEGENDRE FUNCTION:

The associated legendre function p_n^m is defined as

$$p_n^m(x) = (1-x^2)^{+\frac{m}{2}}\frac{d^m}{dx^m}p_n(x)$$

Solution:

Using Rodrigues formula of legendry polynomial

$p_n(x)$=$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$p_n^m(x) = \frac{1}{2^n n!}(1-x^2)^{+\frac{m}{2}}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n$

3.4 ORTHOGONAL PROPERTIES OF LEGENDRY POLYNOMIAL:

$\int_{-1}^{+1} p_n^m(x) \quad p_n^m(x)dx = \frac{2(n+n)!}{(2n+1)(n-m)!} \quad \delta nn'$

Where $\delta nn'$ is called as Kronecker delta function

Where $\delta nn'$ = 0 when n ≠n'

$\delta nn'$=1 when n=n'

Case-I

When n=n'

The associated legendre equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left(n(n+1)-\frac{m^2}{1-x^2}\right)y=0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\left(n(n+1)-\frac{m^2}{1-x^2}\right)y=0 \qquad \ldots\ldots\ldots.. 1$$

Let $p_n^m(x)$ *and* $p_{n'}^m(x)$are the two solutions of associated Legendre polynomial

Then

$$\frac{d}{dx}\left\{(1-x^2)\frac{dp_n^m(x)}{dx}\right\}+\left(n(n+1)-\frac{m^2}{1-x^2}\right)p_n^m(x)=0 \qquad \ldots\ldots\ldots..2$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{dp_{n'}^m(x)}{dx}\right\}+\left(n'(n'+1)-\frac{m^2}{1-x^2}\right)p_{n'}^m(x)=0 \qquad \ldots\ldots\ldots..3$$

Multiplying equation 2 with $p_{n'}^m(x)$ and equation 3 with $p_n^m(x)$ *then*

$$p_{n'}^m(x)\frac{d}{dx}\left\{(1-x^2)\frac{dp_n^m(x)}{dx}\right\}+p_{n'}^m(x)\left(n(n+1)-\frac{m^2}{1-x^2}\right)p_n^m(x)=0$$

$$p_n^m(x)\frac{d}{dx}\left\{(1-x^2)\frac{dp_{n'}^m(x)}{dx}\right\}+p_n^m(x)\left(n'(n'+1)-\frac{m^2}{1-x^2}\right)p_{n'}^m(x)=0$$

Subtract above values and integrate with respect to ‘x’ between the limits -1 and +1 then

$$\int_{-1}^{+1}p_n^m(x)\frac{d}{dx}\left[(1-x^2)\frac{dp_n^m(x)}{dx}\right]dx-\int_{-1}^{+1}p_n^m(x)\frac{d}{dx}(1-x^2)\frac{dp_{n'}^m(x)}{dx}dx$$
$$+\int_{-1}^{+1}[n(n+1)-\frac{m^2}{1-x^2}-n'(n'+1)-\frac{m^2}{1-x^2}]\,p_n^m(x)p_{n'}^m(x)=0$$

$$\left\{p_{n'}^{m}(x)(1-x^2)\frac{dp_n^m(x)}{dx}\right\}_{-1}^{+1} - \int_{-1}^{+1}(1-x^2)\frac{dp_n^m(x)}{dx}\frac{dp_{n'}^m(x)}{dx}dx$$

$$-\left[p_n^m(x)\left\{(1-x^2)\frac{dp_{n'}^m(x)}{dx}\right\}_{-1}^{+1} + dx\right.$$

$$= \int_{-1}^{+1}(1-x^2)\frac{dp_{n'}^m(x)}{dx}dx + \int_{-1}^{+1}[n(n+1)-n'(n'+1)]\,p_n^m(x)p_{n'}^m(x)dx$$

$$= 0$$

$$\left\{p_{n'}^{m}(x)(1-x^2)\frac{dp_n^m(x)}{dx}\right\}_{-1}^{+1} - \left[p_n^m(x)\left\{(1-x^2)\frac{dp_{n'}^m(x)}{dx}\right\}_{-1}^{+1}\right.$$

$$+\int_{-1}^{+1}[n(n+1)-n'(n'+1)]\,p_n^m(x)p_{n'}^m(x)dx = 0$$

Applying limits

$$\int_{-1}^{+1}[n(n+1)-n'(n'+1)]\,p_n^m(x)p_{n'}^m(x)dx = 0$$

$$[n(n+1)-n'(n'+1)\int_{-1}^{+1}p_n^m(x)p_{n'}^m(x)dx = 0$$

$\int_{-1}^{+1} p_n^m(x)p_{n'}^m(x)dx = 0$ when n≠n'

Case-II

When n=n'

$$\int_{-1}^{+1}p_n^m(x)p_{n'}^m(x)dx = \int_{-1}^{+1\grave{}}\{p_n^m(x)\}^2\,dx$$

We know that

$$p_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}p_n(x)$$

$$\int_{-1}^{+1} \left(p_n^m(x)\right)^2 dx = \int_{-1}^{+1} \left\{(1-x^2)^m \frac{d^m}{dx^m} p_n(x)\right\} \frac{d^m}{dx^m} p_n(x) dx$$

Now integrate by parts formula

$$\left\{(1-x^2)^m \frac{d^m}{dx^m} p_n(x) p_n(x)\right\}_{-1}^{+1} - \int_{-1}^{+1} \frac{d^{m-1}}{dx^{m-1}} p_n(x) \frac{d}{dx}\left[(1-x^2)^m \frac{d^m}{dx^m} p_n(x) p_n(x)\right] dx$$

$$\int_{-1}^{+1} \left(p_n^m(x)\right)^2 dx = -\int_{-1}^{+1} \frac{d^{m-1}}{dx^{m-1}} p_n(x) \left[(1-x^2)^m \frac{d^m}{dx^m} p_n(x) p_n(x)\right] dx \quad \ldots\ldots\ldots 4$$

Now, we consider the $p_n(x)$legendre differential equation then we have

$$(1-x^2)\frac{d^2}{dx^2} p_n(x) - 2x \frac{d}{dx} p_n(x) + n(n+1) p_n(x) = 0$$

Differentiate above equation with respect to(m-1) times using Leibentz theorem

$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}} p_n(x) + (m-1)c_1(-2x)\frac{d^m}{dx^m} p_n(x) + (m-1)c_2(-2)\frac{d^{m-1}}{dx^{m-1}} p_n(x) = 0$$

$$(-2x)\frac{d^m}{dx^m} p_n(x) - 2(m-1)c_1 \frac{d^{m-1}}{dx^{m-1}} + n(n+1)\frac{d^{m-1}}{dx^{m-1}} p_n(x) = 0$$

$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}} p_n(x) - 2(m-1)x \frac{d^m}{dx^m} p_n(x) - \frac{2(m-1)(m-2)}{2!}\frac{d^{m-1}}{dx^{m-1}} p_n(x)$$
$$- 2x \frac{d^m}{dx^m} p_n(x) - 2(m-1)\frac{d^{m-1}}{dx^{m-1}} p_n(x) + n(n+1)\frac{d^{m-1}}{dx^{m-1}} p_n(x) = 0$$

Multiply above equation with $(1-x^2)^{m-1}$

$$(1-x^2)^m \frac{d^{m+1}}{dx^{m+1}} p_n(x) - 2mx(1-x^2)^{m-1}\frac{d^m}{dx^m} p_n(x) + \{n(n+1)$$
$$- m(m-1)\}(1-x^2)^{m-1}\frac{d^{m-1}}{dx^{m-1}} p_n(x) = 0$$

$$\frac{d}{dx}\left\{(1-x^2)^m \frac{d^m}{dx^m} p_n(x)\right\} = -\{n(n+1) - m(m-1)\}(1-x^2)^{m-1}\frac{d^{m-1}}{dx^{m-1}} p_n(x) = 0$$

Sub above in equation 4

$$\int_{-1}^{+1} \left(p_n^m(x)\right)^2 dx = \int_{-1}^{+1} \frac{d^{m-1}}{dx^{m-1}} p_n(x) \{n(n+1) - m(m-1)\}(1-x^2)^{m-1} \frac{d^{m-1}}{dx^{m-1}} p_n(x) dx$$

$$= \{n(n+1) - m(m-1)\} \int_{-1}^{+1} \{(1-x^2)^{m-1} \frac{d^{m-1}}{dx^{m-1}} p_n(x)\}^2 dx$$

$$=\{n + m(n-m+1)\} \int_{-1}^{+1} \{p_n^{m-1}(x)\}^2$$

$$p_n^m(x) = [(1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} p_n(x)$$

$$(p_n^m(x))^2 = (1-x^2)^m \{\frac{d^m}{dx^m} p_n(x)\}^2$$

$$(p_n^{m-1}(x))^2 = (1-x^2)^{m-1} \{\frac{d^{m-1}}{dx^{m-1}} p_n(x)\}^2$$

Repeating the same process again 'n' terms

We have

$$\{(n+m)(n-m+1)\} \int_{-1}^{+1} \{p_n^{m-1}(x)\}^2 dx$$

$$\int_{-1}^{+1} \{p_n^m(x)\}^2 dx$$

$$= [(n+m)(n-m+1)][(n+m-1)(n-m-2) \ldots . . (n + 1)n \int_{-1}^{+1} \{p_n(x)\}^2 dx$$

$$\int_{-1}^{+1} \{p_n^m(x)\}^2 dx$$

$$= [(n+m)(n+m-1) \ldots \ldots [(n+1)n(n-1) \ldots \ldots . (n-m+2)(n-m + 1) \int_{-1}^{+1} \{p_n(x)\}^2 dx$$

R.H.S is multiplying and divided with (n-1)!

$$\int_{-1}^{+1} \{p_n^m(x)\}^2 dx = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1} \quad \ldots\ldots\ldots..5$$

Now, if m>0 say m=-l when l>0

$$\int_{-1}^{+1} \{p_n^m(x)\}^2 dx = \int_{-1}^{+1} \{p_n^{-l}(x)\}^2 dx$$

We know that

$$\{p_n^{-m}(x)\} = (-1)^m \frac{(n-m)!}{(n+m)!} p_n^m(x)$$

$$\int_{-1}^{+1} \{p_n^{-l}(x)\}^2 dx = \int_{-1}^{+1} \{(-1)^l \frac{(n-l)!}{(n+l)!}\}^2 [p_n^l(x)] = \{\frac{(n-l)!}{(n+l)!}\}^2$$

$$= \{\frac{(n-l)!}{(n+l)!}\}^2 \int_{-1}^{+1} \{p_n^{-l}(x)\}^2 dx$$

$$= \int_{-1}^{+1} \{\frac{(n-l)!}{(n+l)!}\}^2 \frac{(n+l)!}{(n-l)!}\frac{2}{2n+1} - \int_{-1}^{+1} \{p_n^{-l}(x)\}^2 dx = \frac{2(n-l)!}{(n+l)!\,(2n+1)}$$

Since m=-1

$$\int_{-1}^{+1} \{p_n^{-l}(x)\}^2 dx = \frac{2(n+m)}{(2n+l)!\,(n-m)!}$$

$$\int_{-1}^{+1} \{p_n^{-l}(x)\}^2 dx = \frac{2(n-l)!}{(n+l)!\,(2n+1)}$$

3.5 SUMMARY:

Associated Legendre functions, denoted as$p_l^m(x)$, are solutions to the associated Legendre differential equation and play a crucial role in various fields, including physics and engineering.

3.6 KEYWORDS:

- **Legendre Polynomials ($p_l(x)$:** Solutions to the Legendre differential equation, a special case of the associated Legendre equation when m=0m = 0m=0.
- **Associated Legendre Functions $p_l^m(x)$:** General solutions to the associated Legendre equation, which can be expressed in terms of Legendre polynomials and their derivatives.
- **Ordinary Differential Equation (ODE):** A differential equation involving functions of a single variable and their derivatives.
- **Orthogonality:** A property indicating that the inner product of two functions is zero, implying they are independent in the function space.

3.7 SELF ASSESSMENT:

1) Explain and proof the Associated legendre equation?
2) Explain briefly about orthogonal properties of associated legendre functions

3.8 SUGGESTED READINGS:

1) Handbook of Mathematical Functions

Edited by M. Abramowitz and I.A. Stegun, this comprehensive reference provides detailed information on Legendre functions, including their properties and applications.

2) Mathematical Methods for Physicists

Authored by George B. Arfken and Hans J. Weber, this textbook offers a thorough exploration of Legendre functions within the context of mathematical methods used in physics.

3) A Course of Modern Analysis

Written by E.T. Whittaker and G.N. Watson, this classic text delves into the theory of functions, including a comprehensive treatment of Legendre functions.

Prof. R.V.S.S.N. Ravi Kumar

LESSON-4

BESSEL'S DIFFERENTIAL EQUATION

4.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Bessel's differential equation. The chapter began with understanding of The Power series Solutions, Bessel Functions of the first and second kind, Generating Functions, Orthogonal Properties, Recurrence Relations, Physical Application. After completing this chapter, the student will understand the complete idea about Bessel's Differential Equation.

STRUCTURE:

4.1 INTRODUCTION:

Bessel's differential equation is a fundamental second-order linear ordinary differential equation that appears in various physical and engineering contexts, particularly those involving cylindrical or spherical symmetry. It is expressed as:

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

This equation is named after the German mathematician Friedrich Wilhelm Bessel, who conducted an in-depth study of its solutions in 1824. The solutions, termed Bessel functions, are crucial in solving problems related to wave propagation, static potentials, and heat conduction in cylindrical or spherical geometries. For instance, they are instrumental in analyzing electromagnetic waves in cylindrical waveguides and determining the modes of vibration of circular membranes.

4.2 POWER SERIES SOLUTION OF BESSEL'S FORMULA:

The differential equation of bessel's polynomial is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad \ldots\ldots..(1)$$

Or

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Solution:

Let the solution of equation (1) may be assume in power series method

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \ldots\ldots\ldots(2)$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} \quad \ldots\ldots.(3)$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2} \quad \ldots\ldots(4)$$

Substitute above values in equation (1)

$$\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2} + \frac{1}{x}\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} + \left(1 - \frac{n^2}{x^2}\right)\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2} + \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-2} + \sum_{r=0}^{\infty} a_r x^{k+r} - n^2\sum_{r=0}^{\infty} a_r x^{k+r-2}$$
$$= 0$$

$$\sum_{r=0}^{\infty} a_r[k+r)(k+r-1)+(k+r)-n^2]\,x^{k+r-2}+\sum_{r=0}^{\infty} a_r x^{k+r}=0$$

$$\sum_{r=0}^{\infty} a_r x^{k+r-2}\left[[k+r)(k+r-1+1)-n^2\right]+\sum_{r=0}^{\infty} a_r x^{k+r-2}=0$$

$$\sum_{r=0}^{\infty} a_r x^{k+r-2}\left[(k+r)^2-n^2\right]+\sum_{r=0}^{\infty} a_r x^{k+r}=0 \qquad \ldots\ldots\ldots(5)$$

Equation (5) being an identity we can equating to zero the co-efficient of various powers of 'x'

Let us first equating to the zero the lowest power of 'x'

i.ex^{k-2} (putting r=0)

$a_0(k^2-n^2)=0$

$a_0 \neq 0$ (this is the first term of the series expansion)

$k^2-n^2=0$

(k + n)(k - n)=0

k=-*n*, *k*=*n*

$k=\pm n$

Again equating to zero the coefficient of x^{k-1} by putting *r=0*

$[(k+1)^2-n^2]=0$

$a_1=0$ and

$[(k+1)^2-n^2]\neq 0 \qquad (k=\pm n)$

Now equating to zero the co-efficient of general terms i.ex^{k+r}

r=r+2

$a_{r+2}((k+r+2)^2 - n^2) + a_r = 0$

$$a_{r+2} = \frac{-a_r}{(k+r+2+n)(k+r+2-n)} \quad \ldots\ldots..(6)$$

Put r=1

$a_3 = 0$

Similarly $a_{1=}a_{3=}a_5 = a_7 \ldots\ldots\ldots\ldots. = 0(each)$

Now two cases are arise

Case (i):

When k=n

Then equation (6) becomes

$$a_{r+2} = \frac{-a_r}{(r+2+2n)(r+2)}$$

For $r=0$

$$a_2 = \frac{-a_0}{(2n+2)2} = \frac{-a_0}{2^2(n+1)1!}$$

For $r=2$

$$a_4 = \frac{-a_2}{(4n+4)4} = \frac{-a_2}{2^3(n+2)}$$

$$a_4 = \frac{a_0}{2^4 2!\,(n+1)(n+2)}$$

Now equation (2) becomes

$$y = \sum_{r=0}^{\infty} a_r x^{n+r}$$

$$y = a_0x^n + a_1x^{n+1} + a_2x^{n+2} + a_3x^{n+3} + a_4x^{n+4} + \cdots \ldots ....$$

Substitute above values in equation

$$y = a_0x^n - \frac{a_0}{2^2(n+1)1!}x^{n+1} + \frac{a_0}{2^4 2!\,(n+1)(n+2)}x^{n+4} + \cdots \ldots \ldots$$

$$y = a_0[x^n - \frac{x^{n+2}}{2^2 1!\,(n+1)} + \frac{x^{n+4}}{2^4 2!\,(n+1)(n+2)} \ldots \ldots \ldots ..$$

If $a_0 = \frac{1}{2^n n!}$ then then above equation is called Bessel's first kind solution and is denoted by $J_{\mathrm{n}}(x)$

$$J_{\mathrm{n}}(x) = \frac{1}{2^n n!}[x^n - \frac{x^{n+2}}{2^2 1!(n+1)} + \frac{x^{n+4}}{2^4 2!(n+1)(n+2)} \ldots \ldots \ldots ..$$

$$J_{\mathrm{n}}(x) = \frac{1}{n!}(\frac{x}{2})^n + \frac{(-1)\left(\frac{x}{2}\right)^{n+2}}{1!\,(n+1)!} + \frac{(-1)^2\left(\frac{x}{2}\right)^{n+4}}{2!\,(n+2)!} + \cdots \ldots \ldots$$

$$J_{\mathrm{n}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

Or

$$J_{\mathrm{n}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$\{(n+r)! = \Gamma(n+r+1)\}$$

Case (ii)

When $k=-n$

The replace solution be obtained by replace '*n*' by "-*n*' in the value of $J_{\mathrm{n}}(x)$

$$J_{-\mathrm{n}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

The total solution of differential equation is

$y = AJ_{\mathrm{n}}(x) + BJ_{-\mathrm{n}}(x)$

Where A and B are orbitary constants

4.3 BESSEL FUNCTION OF FIRST AND SECOND KIND:

Bessel functions are fundamental solutions to Bessel's differential equation, which arises in various physical contexts exhibiting cylindrical symmetry. The two primary types are the **Bessel functions of the first kind** $\{J_{\nu}(x)\}$ and the **Bessel functions of the second kind**$\{Y_{\nu}(x)\}$.

Bessel Functions of the First Kind $\{J_{\nu}(x)\}$:

These functions are finite at the origin (x=0) for all real values of the order ν. They are defined by the power series:

$$J_{\nu}(x) = \sum_{r=0}^{\infty}(-1)^k \frac{1}{k!\,\Gamma(k+v+1)}\frac{x}{2}$$

Where Γ denotes the Gamma function.

Bessel Functions of the Second Kind$\{Y_{\nu}(x)\}$:

These functions are singular at the origin and are often used to represent solutions that are singular at x=0. They can be expressed in terms of as$J_{\nu}(x)$:

$$Y_{\nu}(x) = \frac{J_{\nu}(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

For integer values of ν, this expression requires a limiting process due to the singularity at x=0.

4.3 GENERATING FUNCTION:

$$e^{\frac{x(z-\frac{1}{z})}{2}} = \sum_{n=0}^{\infty} z^n J_n(x)$$

$$=(-1)^n \sum_{n=0}^{\infty} z^n J_n J_{+n}(x)$$

$$e^{\frac{x(z-\frac{1}{z})}{2}} = e^{\frac{xz}{2}} e^{\frac{-xz}{2z}}$$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^{\frac{x(z-\frac{1}{z})}{2}} = [1 + \frac{xz/2}{1!} + \frac{(xz/2)^2}{2!} + \cdots + \frac{(xz/2)^{n-1}}{(n-1)!} + \frac{(xz/2)^n}{n!} +]$$

$$[1 + (-1)\frac{(-1)x/2z}{1!} + (-1)^2\frac{(x/2z)^2}{2!} + \cdots + (-1)^{n-1}\frac{(x/2z)^{n-1}}{(n-1)!} + (-1)^n\frac{(x/2z)^n}{n!} +$$

$$(-1)^{n+1}\frac{(x/2z)^{n+1}}{(n+1)!} + \cdots]$$

Collecting the co-efficient of z^n in the above expression

$$(-1)^{n-1}\frac{(x/2)^{n-1}}{2!(n-1)!} + (-1)\frac{(x/2)^n}{n!} + (-1)^n\frac{(x/2)^2}{n!1!} + \frac{(x/2)^n}{n!} - \frac{(x/2)^{n+2}}{(n+1)!} + \frac{(x/2)^{n+4}}{2!(n+2)!} + \ldots\ldots$$

$$\frac{(x/2)^n}{n!} + \frac{(x/2)^{n+2}}{(n+1)!} + (-1)^2\frac{(x/2z)^{n+4}}{2!(n+1)!} + \cdots$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{n+2r}}{r!(n+r)!} = J_n(x)$$

$$e^{\frac{x(z-\frac{1}{z})}{2}} = \sum_{n=0}^{\infty} z^n J_n(x)$$

Now collecting the co-efficient of z^{-n} in the above equation

$$(-1)^n\frac{(x/2)^n}{n!} + (-1)^{n+1}\frac{(x/2)^{n+2}}{1!(n+1)!} + (-1)^{n+2}\frac{(x/2)^{n+4}}{2!(n+2)!} + \cdots\ldots..$$

$$(-1)^n\{\frac{(x/2)^n}{n!} - \frac{(x/2)^{n+2}}{1!(n+1)!} + \frac{(x/2)^{n+4}}{2!(n+2)!} + \cdots\ldots..\}$$

$$(-1)^n\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{n+2r}}{r!(n+r)!}$$

$(-1)^r J_n(x)$

$$e^{\frac{x(z-\frac{1}{z})}{2}} = (-1)^n \sum_{n=0}^{\infty} z^n J_n J_{+n}(x)$$

4.5 ORTHOGONAL PROPERTIES OF BESSELS'S POLYNOMIAL:

If α and β are two different roots of an equation

$J_{n(\mu)}$=0 then orthogonal property of bessel's function is

$$\int_0^1 J_n(\alpha x) J_n(\beta x) x \, dx = 0$$

Proof:

From the differential equation of bessel's polynomial

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \quad \dots \dots \dots (1)$$

Let $J_n(\alpha x)$,$J_n(\beta x)$ are the two solutions of above differential equation then replace y by 'u' and x by 'αx' in equation 1

$$\frac{d^2u}{d(\alpha x)^2} + \frac{1}{\alpha x}\frac{du}{d\alpha x} + \left(1 - \frac{n^2}{(\alpha x)^2}\right) u = 0$$

$$\frac{1}{\alpha^2}\frac{d^2u}{dx^2} + \frac{1}{\alpha^2 x}\frac{du}{dx} + \left(1 - \frac{n^2}{\alpha^2 x^2}\right) u = 0$$

Multiplying above equation with '$\alpha^2 x^2$'

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (\alpha^2 x^2 - n^2) u = 0 \quad \dots \dots \dots (2)$$

Replace 'y' by v and x by βx in equation (1)

$$\frac{d^2v}{d(\beta x)^2} + \frac{1}{\beta x}\frac{dv}{d(\beta x)} + \left(1 - \frac{n^2}{(\beta x)^2}\right) v = 0$$

$$\frac{1}{\beta^2}\frac{d^2v}{dx^2} + \frac{1}{\beta^2 x}\frac{dv}{dx} + \left(1 - \frac{n^2}{(\beta x)^2}\right)v = 0$$

Multiplying above equation with $\beta^2 x^2$

$$x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx} + (\beta^2x^2 - n^2)v = 0 \qquad \dots\dots\dots\dots (3)$$

Multiplying equation (2) with $\frac{v}{x}$ $equation$ (3)$with$ $\frac{u}{x}$

$$xv\frac{d^2u}{dx^2} + v\frac{du}{dx} + (\alpha^2x^2 - n^2)\frac{uv}{x} = 0 \qquad \dots\dots\dots.(4)$$

$$xu\frac{d^2v}{dx^2} + u\frac{dv}{dx} + (\beta^2x^2 - n^2)\frac{uv}{x} = 0 \qquad \dots\dots\dots..(5)$$

Subtract equation (5) from equation (4)

$$x\left[v\frac{d^2u}{dx^2} - u\frac{d^2v}{dx^2}\right] + \left[v\frac{du}{dx} - u\frac{dv}{dx}\right] + x^2(\alpha^2 - x^2)\frac{uv}{x}x = 0$$

$$x\left[v\frac{d^2u}{dx^2} - u\frac{d^2v}{dx^2}\right] + \left[v\frac{du}{dx} - u\frac{dv}{dx}\right] + (\alpha^2 - x^2)uvx = 0$$

Integrating above equation with limits 0 to 1

$$\int_0^1 \frac{d}{dx}\left\{x\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]\right\}dx + (\alpha^2 - x^2)\int_0^1 uvx\,dx = 0$$

$$x\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]_0^1 + (\alpha^2 - x^2)\int_0^1 uvx\,dx = 0$$

$$\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]_0^1 + (\alpha^2 - x^2)\int_0^1 uvx\,dx = 0$$

$$[J_n(\beta x)J_n'(\alpha x)\alpha - [J_n(\alpha x)\beta J_n'(\beta x)]_0^1 + (\alpha^2 - x^2)\int_0^1 J_n(\alpha x)J_n(\beta x)x\,dx = 0$$

$$[J_n(\beta)J_n'(\alpha x)\alpha - [J_n(\alpha)\beta J_n'(\beta x)]_0^1 + (\alpha^2 - x^2)\int_0^1 J_n(\alpha x)J_n(\beta x)x\,dx = 0$$

If α and β are two different roots (i.e α≠β) of an equation $J_n(\mu) = 0$

i.e$J_n(\alpha) = J_n(\beta) = 0$

$$\int_0^1 J_n(\alpha x)J_n(\beta x)x\,dx + 0 = 0$$

$$\int_0^1 J_n(\alpha x)J_n(\beta x)x\,dx = 0$$

4.6 RECURRENCE RELATIONS OF BESSEL'S DIFFERENTIAL EQAUTAION:

1)$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

2)$xJ_n'(x) = -nJ_n(x) + xJ_{n+1}(x)$

3)$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

4)$2nJ_n(x) = x[J_{n-1}(x) + xJ_{n+1}(x)]$

5)$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

6)$\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$

First Recurrence relation:

$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

We know that

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{n+2r}}{r!\,(n+r)!}$$

Differentiate with respect to 'x' on both sides

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)({}^{x}/_{2})^{n+2r-1}\frac{1}{2}}{r!\,(n+r)!}$$

Multiplying with 'x' on both sides

$$xJ_n'(x) = n\sum_{r=0}^{\infty} \frac{(-1)^r ({}^{x}/_{2})^{n+2r}}{r!\,(n+r)!} + x\sum_{r=0}^{\infty} \frac{(-1)^r ({}^{x}/_{2})^{n-1+2r}}{(r-1)!\,(n+r)!}$$

Let r-1=s

$$xJ_n'(x) = nJ_n(x) + x\sum_{s=0}^{\infty} \frac{(-1)^{s+1} ({}^{x}/_{2})^{n-1+2s}}{s!\,(n+1+s)!}$$

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Second recurrence relation:

$$xJ_n'(x) = -nJ_n(x) + xJ_{n+1}(x)$$

Proof:

We know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r ({}^{x}/_{2})^{n+2r}}{r!\,(n+r)!}$$

Differentiate with respect to 'x' on both sides

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)({}^{x}/_{2})^{n+2r-1}\frac{1}{2}}{r!\,(n+r)!}$$

Multiplying with 'x' on both sides

$$xJ_n'(x) = x\sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-n)({}^{x}/_{2})^{n+2r-1}\frac{1}{2}}{r!\,(n+r)!}$$

$$xJ_n'(x) = -n\sum_{r=0}^{\infty} \frac{(-1)^r ({}^{x}/_{2})^{n+2r}}{r!\,(n+r)!} + x\sum_{r=0}^{\infty} \frac{(-1)^{s+1} ({}^{x}/_{2})^{n-1+2r}}{s!\,(n-1+r)!}$$

$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$

Third recurrence relation:

$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

We know that first recurrence relation

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \quad \ldots\ldots\ldots.(1)$$

From second recurrence relation

$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \quad \ldots\ldots\ldots(2)$

Adding (1) and (2)

$xJ_n'(x) + xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) - nJ_n(x) + xJ_{n-1}(x)$

$2xJ_n'(x).x = x[J_{n-1}(x) - J_{n+1}(x)]$

$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

Fourth Recurrence relation:

$2nJ_n(x) = x[J_{n-1}(x) + xJ_{n+1}(x)]$

We know that first recurrence relation

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \quad \ldots\ldots\ldots.(1)$$

From second recurrence relation

$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \quad \ldots\ldots\ldots(2)$

Subtract (1) and (2) then

$xJ_n'(x) - xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) + nJ_n(x) - xJ_{n-1}(x)$

$2nJ_n(x) - xJ_{n+1}(x) - xJ_{n-1}(x) = 0$

$2nJ_n(x) = x[J_{n-1}(x) + xJ_{n+1}(x)]$

Fifth recurrence relation:

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

We know that first recurrence relation

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Multiplying above equation with x^{-n-1} on both sides we have

$$x^{-n}J_n'(x) = nx^{-n-1}J_n(x) - x^{-n}J_{n+1}(x)$$

$$x^{-n}J_n'(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

Sixth Recurrence Relation:

$$\frac{d}{dx}[x^{n}J_n(x)] = x^{n}J_{n-1}(x)$$

From second recurrence relation

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$$

Multiplying above equation with x^{n-1} on both sides we have

$$x^{n}J_n'(x) = -nx^{n-1}J_n(x) + x^{n}J_{n+1}(x)$$

$$x^{n}J_n'(x) + nx^{n-1}J_n(x) = x^{n}J_{n+1}(x)$$

$$\frac{d}{dx}[x^{n}J_n(x)] = x^{n}J_{n-1}(x)$$

4.7 PHYSICAL APPLICATIONS OF BESSEL'S DIFFERENTIAL EQUATION:

Bessel's differential equation appears in many physical problems, especially those involving cylindrical symmetry. It has significant applications in fields such as acoustics, electromagnetics, fluid dynamics, and heat conduction. Here are a few physical scenarios where Bessel's differential equation arises:

1. Vibration of Circular Membranes (Acoustics):

- **Application**: In the study of vibration of circular membranes, such as drums, Bessel's equation governs the radial displacement of the membrane.
- **Physical System**: If a circular drumhead is stretched and struck, the displacement at any point on the membrane can be described using Bessel functions of the first and second kinds.
- **Reason for Bessel's Equation**: The solution involves cylindrical coordinates, where the radial component of the wave equation leads to Bessel's equation.

2. Heat Conduction in Cylindrical Coordinates

- **Application**: Bessel's equation can describe temperature distribution in a cylindrical object over time.
- **Physical System**: Imagine a long, solid cylinder (like a pipe) with heat applied at one end. The temperature variation with respect to the radius and time can be modeled using Bessel functions.
- **Reason for Bessel's Equation**: The cylindrical symmetry of the problem leads to the use of cylindrical coordinates, and the radial part of the heat equation in such coordinates results in Bessel's differential equation.

3. Electromagnetic Waves in Cylindrical Structures

- **Application**: Bessel's equation is used to describe the behavior of electromagnetic waves in waveguides, especially those with a circular cross-section.
- **Physical System**: In a coaxial cable, fiber-optic cables, or certain types of lasers, the electric and magnetic field distributions can be modeled with Bessel functions.
- **Reason for Bessel's Equation**: The wave equation in cylindrical coordinates has a radial part that results in Bessel’s differential equation, which describes the variation of the fields in the radial direction.

4. Sound Propagation in Cylindrical Pipes

- **Application**: The propagation of sound waves in a cylindrical pipe, such as in acoustical engineering or pipe organ design, often leads to solutions involving Bessel functions.

- **Physical System**: When sound waves are propagating through pipes or tubes with circular cross-sections, the radial displacement of the sound wave is described by Bessel functions.
- **Reason for Bessel's Equation**: The wave equation for sound in cylindrical coordinates leads to the Bessel differential equation, with the radial displacement being a function of Bessel functions.

5. Quantum Mechanics (Particle in a Cylindrical Potential)

- **Application**: In quantum mechanics, when a particle is confined within a cylindrical potential, such as in a cylindrical quantum well, the wavefunctions of the particle often involve Bessel functions.
- **Physical System**: For a particle in a potential that has cylindrical symmetry (like an electron in a cylindrical nanowire), the Schrödinger equation in cylindrical coordinates results in Bessel's equation for the radial part of the wavefunction.
- **Reason for Bessel's Equation**: The separation of variables in the Schrödinger equation leads to the radial equation that is in the form of Bessel's equation.

6. Fluid Flow in Pipes

- **Application**: The flow of fluids in cylindrical pipes can also be described by Bessel's equation when analyzing the radial velocity distribution in steady-state flows.
- **Physical System**: In analyzing the flow of liquids or gases through pipes, especially in the presence of specific boundary conditions (like a pipe with a circular cross-section), Bessel functions often arise in the solution of the governing Navier-Stokes equations.
- **Reason for Bessel's Equation**: The radial component of the velocity field in cylindrical coordinates leads to an equation involving Bessel functions.

4.8 SUMMARY:

Understanding and working with Bessel's differential equation and its solutions and the complete idea about Bessel’s Differential Equation.

4.9 KEY WORDS:

Bessel's differential equation is a second-order linear ordinary differential equation that frequently arises in problems exhibiting cylindrical or spherical symmetry. Key terms associated with this equation include:

- **Bessel functions**: Solutions to Bessel's differential equation, commonly denoted as $J_\alpha(x)$ for the first kind and $Y_\alpha(x)$ for the second kind.
- **Order (α)**: A parameter that defines the specific form of the Bessel function, representing the equation's order.
- **Cylindrical symmetry**: A system characteristic where physical properties are invariant under rotations about a central axis, leading to the appearance of Bessel functions in the solutions.
- **Spherical symmetry**: A system where properties are uniform in all directions from a central point, often resulting in solutions involving spherical Bessel functions.
- **Frobenius method**: A technique used to find power series solutions to differential equations near a singular point, applicable to Bessel's equation.
- **Gamma function (Γ)**: A function extending the factorial to complex numbers, appearing in the series representation of Bessel functions.
- **Neumann function**: Another name for the Bessel function of the second kind, $Y_\alpha(x)$
- **Modified Bessel functions**: Solutions to the modified Bessel's equation, denoted as $I_\alpha(x)$ and $I_\alpha(x)$, relevant for problems involving hyperbolic functions.
- **Recurrence relations**: Formulas expressing Bessel functions of different orders in terms of each other, useful for computational purposes.
- **Orthogonality**: A property indicating that Bessel functions of different orders are orthogonal over a specific interval with a given weight function, important in solving boundary value problems.

These terms are fundamental in understanding and working with Bessel's differential equation and its solutions.

4.10 SELF ASSESSMENTS QUESTIONS:

1) Explain about power series solution of Bessel’s differential equation?
2) Briefly explain about Recurrence relations of Bessel’s equations?
3) Write about the orthogonal properties of Bessel’s equations?

4.11 SUGGESTED READINGS:

For a comprehensive understanding of Bessel's differential equations, the following reference books are highly recommended:

Introduction to Bessel Functions

Authored by Frank Bowman, this book offers a clear introduction to the properties and applications of Bessel functions, covering topics such as Bessel functions of zero order, modified Bessel functions, definite integrals, and asymptotic expansions.

A Treatise on the Theory of Bessel Functions

Written by G.N. Watson, this monumental treatise delves deep into the theory of Bessel functions, providing extensive mathematical insights and is considered a standard reference in the field.

Advanced Mathematics for Applications

This textbook by Thomas J. Pence and Indrek S. Wichman includes a dedicated chapter on the Bessel equation, offering practical applications and detailed explanations suitable for advanced studies

Prof. R.V.S.S.N. Ravi Kumar.

LESSON-5

HERMITE DIFFERENTIAL EQUATION

5.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Hermite differential equation. The chapter began with understanding of The Power series Solutions, Hermite polynomials, Generating Functions, Orthogonality, Recurrence Relations, Rodrigues formula, Physical Applications. After completing this chapter, the student will understand the complete idea about Hermite Differential Equation.

STRUCTURE:

5.1 INTRODUCTION:

Hermite polynomials are the power series solution of second order Hermite differential equation with variable coefficients. They will be need mainly as a mathematical tool in dromy of the scientific problem. Very familiar applications in quantum mechanics is the simple harmonic oscillator. While the ground state is given by the Gaussian function, higher states are given by products of the respective orders of Hermite polynomials with the gause are functions.

5.2 SOLUTION OF HERMITE'S DIFFERENTIAL EQUATION:

This equation is of the form

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2xy = 0 \qquad \text{----------} \quad (1)$$

where y is a parameter.

Suppose its series solution is

$$y = \sum_{r=0}^{\infty} a_r x^{k+r}. \quad a_0 \neq 0 \text{ and k is a constant} \qquad \text{----------} \quad (2)$$

So that $\frac{dy}{dx} = \sum\limits_{r=0}^{\infty} a_r (k + r)\, x^{k+r-1}$

and $\frac{d^2y}{dx^2} = \sum\limits_{r=0}^{\infty} a_r (k + r)(k + r - 1)\, x^{k+r-2}$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get the identity

$$\sum_{r=0}^{\infty} [(k + r)(k + r - 1)\, x^{k+r-2} - 2(k + r - \nu)\, x^{k+r}]a_r \equiv 0 \qquad \text{----------} \quad (3)$$

Equating the Coefficient of the first term (i.e. x^{k-2}) (by putting r = 0, to zero, we get

$a_0\, k(k - 1) = 0$ giving k = 0, 1 as $a_0 \neq 0$ --------- (4)

Now, equating to zero the coefficient of second term (i.e. x^{k-1}) in (3) we get

$a_1\, k\,(k + r) = 0$ i.e. $a_1 = 0$ when k = - 1 and a_1 may or may not be zero when k = 0, as the values of k are ahead fixed as in (4)

Also equating the coefficient of x^{k+r} to zero, we find

$a_{r+2}\,(k + r + 2)(k + r + 1) - 2a_r\,(k + r - \nu) = 0$

giving the recurrence relation

$$a_{r+2} = \frac{2(k + r - v)}{(k + r + 2)(k + r + 1)}\, a_r \qquad \text{----------} \quad (5)$$

when k = 0, (5) becomes $a_{r+2} = \dfrac{2(r - v)}{(r+2)(r+1)} a_r$ ---------- (6)

and when k = 1, (5) becomes $a_{r+2} = \dfrac{2(1+r - v)}{(r+3)(r+2)} a_r$ ---------- (7)

Case-I: When k = 0, putting r = 0, 1, 2, 3,… in (6) we have

$$a_2 = -\frac{2}{\underline{|2}} v a_0; \quad a_3 = -\frac{2(v-1)}{\underline{|3}} a_1$$

$$a_4 = -\frac{2^2 v(v-2)}{\underline{|4}} a_0 \ ; \ a_5 = -\frac{2^2(v-1)(v-3)}{\underline{|5}} a_1 \ \ldots\ldots\ldots\ldots.. \ a_{2r} \frac{(-2)^r v(v-2)\ldots(v-2r+2)}{\underline{|2r}} a_0;$$

$$a_{2r+1} = \frac{(-2)^r (v-1)(v-3)\ldots(v-2r+1)}{\underline{|2r+1}} a_1$$

Now if $a_1 = 0$, then $a_3 = a_5 = a_7 = a_{2r+1} = \ldots = 0$.

But if $a_1 \neq 0$, then (2) gives for k = 0, $y = \sum\limits_{r=1}^{\infty} a_r x^r$

i.e. $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$= a_0 + a_2x^2 + a_4x^4 + \ldots + a_1x + a_3x^3 + a_5x^5 + \ldots$

$$= a_0\left[1 - \frac{2v}{\underline{|2}} x^2 + \frac{2^2 v(v-2)}{\underline{|4}} x^4 - \ldots + (-1)^r \frac{2^r}{\underline{|2r}} v(v-2)\ldots(v-2r+2) x^{2r} + \ldots\right]$$

$$+ a_1 x\left[1 - \frac{2(v-1)}{\underline{|3}} x^2 + \frac{2^2 (v-1)(v-3)}{\underline{|5}} x^4 - \ldots\right.$$

$$\left. + (-1)^r \frac{2^r}{\underline{|2r+1}} (v-1)(v-3)\ldots(v-2r+1) x^{2r} + \ldots\right] \qquad \text{---------- (8)}$$

$$= a_0\left[1 + \sum_{r=1}^{\infty} \frac{(-1)^r 2^r}{\underline{|2r+1}} v(v-2)\ldots(v-2r+2) x^{2r} + \ldots\right]$$

$$+ a_1\left[x + \sum_{r=1}^{\infty} \frac{(-1)^r 2^r}{2r+1}(v-1)(v-3)\ldots(v-2r+1)x^{2r+1}\ldots \right. \quad \text{----------} \ (9)$$

Case II: When k = 1, then $a_1 = 0$ and so by putting r = 0, 1, 2, 3, … in (7)
we find

$$a_2 = -\frac{2(v-1)}{\underline{|3}}\ a_0$$

$$a_4 = \frac{2^2(v-1)(v-3)}{\underline{|5}}\ a_0 \ \ldots\ldots\ldots\ldots\ldots\ldots \ a_{2r} = (-1)\frac{2^r(2v-1)(v-3)\ldots(v-2r+1)}{\underline{|2r+1}}\ a_0$$

Hence the solution is

$$= a_0 x\left[1 - \frac{2(v-1)}{\underline{|3}}x^2 + \frac{2^2(v-1)(v-3)}{\underline{|5}}x^4 - \ldots\right.$$

$$\left. + \frac{(-1)^r 2^r (v-1)(v-3)\ldots(v-2r+1)}{\underline{|2r+1}}x^{2r} + \ldots\right] \quad \text{----------} \ (10)$$

clearly the solution (10) is included in the second part of (8) except that a_0 is replaced by a_1 and hence in order that the Hermite equation may have two independent solutions, a_1 must be zero, even if k = 0 and then (8) reduces to

$$y = a_0\left[1 - \frac{2v}{\underline{|2}}x^2 + \frac{2^2 v(v-2)}{\underline{|4}}x^4 - \ldots + (-1)^r \frac{2^r}{\underline{|2r}}v(v-2)\ldots(v-2r+2)x^{2r} + \ldots\right] \quad \text{----------} \ (11)$$

The complete integral of (1) is then given by

$$y = A\left[1 - \frac{2v}{\underline{|2}}x^2 + \frac{2^2 v(v-2)}{\underline{|4}}x^4 - \ldots\right] + Bx\left[1 - \frac{2(v-1)}{\underline{|3}}x^2 + \frac{2^2(v-1)(v-3)}{\underline{|5}}x^4 - \ldots\right]$$

$$\text{----------} \ (12)$$

where A and B are arbitrary constants.

Where ***v*** is an integer, then the resulting solution is called Hermite Polynomial. The arbitrary constant A and B are taken as $(-1)^{v/2}.\dfrac{\underline{|v}}{\underline{\left|\frac{v}{2}\right.}}$ and $(-1)^{\frac{v-1}{2}}$ respectively.

In equation (12), the series with coefficient A alone is taken as the Hermite Polynomial of **even** order ***v***and that with coefficient B alone is considered as Hermite Polynomial of **odd** order ***v***.

5.3 HERMITE POLYNOMIALS:

The Hermite polynomial $H_n(x)$ is defined as

$$f(x, t) = e^{2tx - t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{\underline{|n}} \quad \text{----------} \quad (13)$$

for all integral values of n and all real values of x. (At a later stage, it will be proved that the exponential function is the **generating function of H_n (x))** (13) can be written as

$$f(x, t) = e^{x^2}\, e^{-(x-t)^2} = \frac{H_0(x)}{\underline{|0}} + \frac{H_1(x)}{\underline{|1}}\, t + \frac{H_2(x)}{\underline{|2}}\, t^2 + \ldots + \frac{H_n(x)}{\underline{|n}}\, t^n + \ldots$$

So that $\left[\dfrac{\partial^n f(x, t)}{\partial t^n}\right]_{t=0} = \dfrac{H_n(x)}{\underline{|n}}\, \underline{|n} = H_n(x)$

$$= \left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0} e^{x^2} \quad \text{----------} \quad (14)$$

If we put x – t = p i.e. t = x – p for t = 0 gives x = p

and $\dfrac{\partial}{\partial t} = -\dfrac{\partial}{\partial p}$ so that $\dfrac{\partial^n}{\partial t^n}\left\{e^{-(x-t)^2}\right\} = (-1)^n \dfrac{\partial^n}{\partial p^n} e^{-p^2}$

$$\therefore \left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0} = (-1)^n \frac{\partial^n}{\partial x^n} e^{-x^2} = (-1)^n \frac{d^n}{dx^n} e^{-x^2} \quad \text{----------} \quad (15)$$

From (14) and (15), we therefore have

$$H_n(x) = e^{x^2} (-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad \text{(Rodrigue's formula)} \quad \text{----------} \quad (16)$$

From (16) we can calculate Hermite polynomials of various degrees such as

$$\left.\begin{array}{ll} H_0(x)=1 & H_4(x)=16x^4-48x^2+12 \\ H_1(x)=2x & H_5(x)=32x^5-160x^3+120x \\ H_2(x)=4x^2-2 & H_6(x)=64x^6-480x^4+720x^2-120 \\ H_3(x)=8x^3-12x & H_7(x)=128x^7-1344x^5+3360x^3-1680x \end{array}\right\} \text{----------}\quad (17)$$

5.4 GENERATING FUNCTIONS:

Q: To prove that $e^{2tx-t^2} = \sum_{r=0}^{\infty} \frac{t^n}{n!} H_n(x)$. Where e^{2tx-t^2} is called the generating function of $H_n(x)$.

Solution: We have

$$e^{2tx-t^2} = e^{2tx}\, e^{-t^2} = \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \cdot \sum_{s=0}^{\infty} \frac{\left(-t^2\right)^s}{s!} \sum_{r,s=0}^{\infty} \frac{(2x)^r}{r!s!} . t^{r+2s}$$

$\therefore$ Coefficient of t^n (for fixed value of s)

$$= (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!s!} \qquad \text{(put } r + 2s = n)$$

But the total coefficient of t^n is obtained by summing over all allowed values of s, (for $r = n - 2s \geq 0$)

$\therefore$ $n - 2s \geq 0$ i.e. $s \leq (n/2)$. So we can say, that if n is even s goes from 0 to (n / 2) and if n is odd, s goes from 0 to (n – 2) / 2.

Hence required coefficient of t^n = $\sum_{s=0}^{[n/2]} (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!s!} = \frac{H_n(x)}{n!}$

(Here $\left[\frac{n}{2}\right]$ means the greatest integer that does not exceed $\frac{n}{2}$).

$$\Rightarrow e^{2tx-t^2} \sum_{n=1}^{\infty} \frac{t^n}{n!} H_n(x) \text{ i.e., } \quad e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H(x).$$

5.5 ORTHOGONAL PROPERTIES OF HERMITE POLYNOMIALS:

Now since $H_n(x)$ is a solution of Hermite equation, we have

$H''_n(x) - 2x\, H'_n(x) + 2n\, H'_n(x) = 0$ by (22)

If we put $y = e^{-x^2/2} H_n(x)$ i.e., $H_n(x) = y\, e^{x^2/2}$

So that $H'_n(x) = y'\, e^{x^2/2} + xy\, e^{x^2/2}$

and $H''_n(x) = y''\, e^{x^2/2} + 2xy'\, e^{x^2/2} + y(1 + x^2)\, e^{x^2/2}$

then we get $y'' + (1 - x^2 + 2n)y = 0$ ---------- (30)

Since $y = e^{-x^2/2} H_n(x) = \psi_n(x)$ by (26), it therefore follows that $\psi_n(x)$ satisfies (30) and hence

$\psi''_n + (2n + 1 - x^2)\, \psi_n = 0$ ---------- (31)

for a function ψ_m, this relation is

$\psi''_m + (2m + 1 - x^2)\, \psi_m = 0$ ----------- (32)

Multiplying (31) by ψ_m**;** (32) by ψ_n and subtracting we get

$2(m - n)\, \psi_m\psi_n = \psi_m\psi''_n - \psi_n\psi''_m$ ---------- (33)

Integrating over $(-\infty, \infty)$, we have

$$2(m-n)\int_{-\infty}^{\infty} \psi_m\psi_n\, dx = \int_{-\infty}^{\infty} (\psi_m\psi''_n - \psi_n\psi''_m)dx$$

$$= \left[\psi_m\psi'_n - \psi_n\psi'_m\right]_{-\infty}^{\infty} (\psi'_m\psi'_n - \psi'_n\psi'_m)dx \qquad \text{(on integrating by parts)}$$

$$= 0$$

$\because \psi_n(x) \to 0$ as $|x| \to \infty$ for all positive integral values of n.

or $\int_{-\infty}^{\infty} \psi_m\psi_n\, dx = 0$ if $m \neq n$

symbolically $I_{m,n} = \int_{-\infty}^{\infty} \psi_m\psi_n\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, H_m(x)\, H_n(x)dx = 0$

when $m \neq n$ ---------- (34)

In particular $I_{n-1,\, n+1} = 0$ ---------- (35)

Now from (28) we have $2x\, \psi_n(x) = 2n\psi_{n-1}(x) + \psi_{n+1}(x)$

$$\therefore \int_{-\infty}^{\infty} 2x\, \psi_n(x)\, \psi_{n-1}\, dx = 2n \int_{-\infty}^{\infty} \psi_{n-1}(x) + \psi_{n+1}(x)$$

$$\because \quad \int_{-\infty}^{\infty} \psi_{n-1} + \psi_{n+1}\, dx = 0 \text{ by (35)}$$

$$= 2n\, I_{n-1,\, n+1} \qquad \text{---------- (36)}$$

Also $\psi_n(x) = e^{-x^2/2} H_n(x)$

$$= (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \text{by (16)}$$

Thus (36) gives

$$-\int_{-\infty}^{\infty} 2x\, e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right) dx = 2n\, I_{n-1,\, n-1}$$

$$\text{or } 2n\, I_{n-1,\, n-1} = -\int_{-\infty}^{\infty} d\,(e^{x^2}) \frac{d^n}{dx^n}\left(e^{-x^2}\right) \frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right) dx$$

$$= \left[e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2}) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{x^2}$$

$$\left\{ \frac{d^n}{dx^n}\left(e^{-x^2}\right) \frac{d^n}{dx^n}\left(e^{-x^2}\right) \frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) \frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right) \right\} dx \qquad \text{(on integrating by parts)}$$

$$= 0 + I_{n,\, n} + I_{n+1,\, n-1}$$

$$= I_{n,\, n} \quad \text{by (35)}$$

$$\therefore\ I_{n,\, n} = 2n\, I_{n-1,\, n-1} \qquad \text{---------- (37)}$$

Applying (37), repeatedly, we have

$I_{n,\,n} = 2n\, I_{n-1,\,n-1} = 2n\, 2(n-1)\, I_{n-2,\,n-2}$

$= 2^2\, n(n-1).2(n-2)\, I_{n-3,\,n-3}$

$= 2^3\, n(n-1)\,(n-2)\, I_{n-3,\,n-3}$

$= \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$

$= 2^n\, n(n-1)\,(n-2) \ldots\ldots 3.2.1.\ I_{0,\,0}$

where $I_{0,\,0} = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ (From Beta and Gamma functions)

$\therefore\ I_{n,\,n} = 2n\, \underline{|n}\, \sqrt{\pi}$ ---------- (38)

Combining the two results (34) and (38); we have in terms of Kronecker delta symbol

$I_{m,\,n} = \int_{-\infty}^{\infty} e^{-x^2}\, H_m(x)\, H_n(x)\, dx = 2^n \underline{|n}\, \sqrt{\pi}\, \delta_{m,\,n}$ ---------- (39)

Where $\delta_{m,\,n} = 0$ when $m \neq n$

$= 1$ when $m = n$.

(39) may also be written as

$I_{m,\,n} = \int_{-\infty}^{\infty} \psi_m(x)\, \psi_n(x)\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, H_m(x)\, H_n(x)\, dx$

$= 2^n \underline{|n}\, \sqrt{\pi}\, \delta_{m,\,n}$ ---------- (40)

Again $2x\, \psi_n(x) = 2n\, \psi_{n-1}(x) + \psi_{n+1}(x)$ gives

$\int_{-\infty}^{\infty} x\, \psi_m(x)\, \psi_n(x)\, dx = nI_{m,\,n-1} + \frac{1}{2} I_{m,\,n+1}$

$= 0$ for $m \neq n = 1$

and $\int_{-\infty}^{\infty} x\, \psi_n \psi_{n+1}(x) dx = n\, I_{n+1,\,n-1} + \frac{1}{2}\, I_{n+1,\,n-1}$

$$= \frac{1}{2} 2^{n+1} \underline{|n+1} \sqrt{\pi} \text{ as above}$$

$$= 2^n \underline{|(n+1)} \sqrt{\pi} \quad \text{for } m = n$$

Hence $\int_{-\infty}^{\infty} x\, \psi_m(x)\, \psi_n(x)\, dx = 2^n \underline{|n+1} \sqrt{\pi}\, \delta_{m,\,n}$ ---------- (41)

Further $2n\, \psi_{n-1}(x) = x\, \psi_n(x) + \psi'_n(x)$ gives

$$\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2n \int_{-\infty}^{\infty} \psi_m(x)\, \psi_{n-1}(x)\, dx - \int_{-\infty}^{\infty} x\, \psi_m(x)\, \psi_n(x)\, dx$$

$$= 0 \text{ if } m \neq n = 1$$

and $2n\, I_{n-1,\, n-1} - 2^{n-1} \underline{|n} \sqrt{\pi}$ if $m = n = 1$

$$= 2^n \underline{|n} \sqrt{\pi} - 2^{n-1} \underline{|n} \sqrt{\pi} = 2^{n-1} \underline{|n} \sqrt{\pi}$$

Hence $\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2^{n-1} \underline{|n} \sqrt{\pi}\, \delta_{m,\,n}$ ---------- (42)

In the last if we take $m = n + 1$, then

$$\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2n \int_{-\infty}^{\infty} \psi_{n+1}(x)\, \psi_{n-1}(x)\, dx - \int_{-\infty}^{\infty} \psi_{n+1}(x)\, \psi_n(x)\, dx$$

$$= -2^{n-1} \underline{|n} \sqrt{\pi}\, .$$

Q: Prove that $H_n(-x) = (-1)^n H_n(x)$

Solution: We have $\sum_{n=0}^{\infty} \frac{H_n(x) t^n}{\underline{|n}} = e^{2tx - t^2} = e^{2tx}\, e^{t^2} = \sum_{n=0}^{\infty} \frac{(2x)^n t^n}{\underline{|n}} \times \sum_{n=0}^{\infty} \frac{(-1) t^{2n}}{\underline{|n}}$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n/2} \frac{(-1)^k (2x)^{n-2k}}{\underline{|k}\, \underline{|n-2k}} t^n.$$

Equating coefficient of $\frac{t^n}{\underline{|n}}$ on either side, we get

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \underline{|n}\,(2x)^{n-2k}}{\underline{|k}\,\underline{|n-2k}}$$

Replacing x by $-x$ we get

$$H_n(-x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \underline{|n}\,(-2x)^{n-2k}}{\underline{|k}\,\underline{|n-2k}}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (-1)^{n-2k} \underline{|n}\,(2x)^{n-2k}}{\underline{|k}\,\underline{|n-2k}}$$

$$= (-1)^n \sum_{k=0}^{[n/2]} \frac{(-1)^k \underline{|n}\,(2x)^{n-2k}}{\underline{|k}\,\underline{|n-2k}}$$

$$= (-1)^n H_n(x)$$

Q: Prove $\int_{-\infty}^{\infty} x\, e^{-x^2} H_n(x)\, H_m(x)\, dx = \sqrt{\pi}\, [2^{n-1} \underline{|n}\, \delta_{m,\, n-1} + 2^n \underline{|n+1}\, \delta_{n+1,\, m}]$

Solution: Integrating by parts we have

$$\int_{-\infty}^{\infty} x\, e^{-x^2} H_n(x)\, H_m(x)\, dx = \left.\int -\frac{1}{2} e^{-x^2} H_n(x)\, H_m(x)\, dx\right]_{-\infty}^{\infty}$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} \frac{d}{dx} \{H_n(x)\, H_m(x)\}\, dx$$

$$= 0 + \int_{-\infty}^{\infty} e^{-x^2} \{H'_n(x)\, H_m(x) + H_n(x)\, H'_m(x)\}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} [2n\, H_{n-1}(x)\, H_m(x) + 2m\, H_n(x)\, H_{m-1}(x)\, dx]\, dx$$

by (20)

$$= n \int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x)\, H_m(x)\, dx + \int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_{m-1}(x)\, dx$$

$$= n\, \sqrt{\pi}\, 2^{n-1} \underline{|n-1}\, \delta_{m,\, n-1} + m\, \sqrt{\pi}\, 2^n \underline{|n}\, \delta_{n,\, m-1}$$

(by orthogonal properties)

$$= \sqrt{\pi}\,[2^{n-1}\,\underline{|n}\,\delta_{m,\,n-1} + 2^{n}\,\underline{|n+1}\,\delta_{n+1,\,m}]$$

$\because \delta_{n,\,m-1} = \delta_{n+1,\,m}$.

5.6 RECURRENCE RELATIONS:

1) $H_n'(x) = 2n\,H_{n-1}(x)$

2) $2x\,H_n(x) = 2n\,H_{n-1}(x) + H_{n+1}(x)$

3) $H_n'(x) = 2x\,H_n(x) - H_{n+1}(x)$

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!}$$

$$e^{2tx-t^2}\,.\,2t = \sum_{n=0}^{\infty} \frac{t^n H_n{'}(x)}{n!}$$

$$2t\,.\sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!} = \sum_{n=0}^{\infty} \frac{t^n H_n{'}(x)}{n!}$$

Comparing the coefficients of t^n on both sides

$$\frac{2\,H_{n-1}(x)}{(n-1)!} = \frac{H_n{'}(x)}{n!}$$

$$\frac{2\,H_{n-1}(x)}{(n-1)!} = \frac{H_n{'}(x)}{n(n-1)!}$$

$$2n\,H_{n-1}(x) = H_n{'}(x)$$

$$H_n{'}(x) = 2n\,H_{n-1}(x)$$

1) STATEMENT: $2x\,H_n(x) = 2n\,H_{n-1}(x) + H_{n+1}(x)$

Proof: Consider Generating Function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!}$$

Differentiate the above equation with respect to t on both sides

$$e^{2tx-t^2}\,(2x-2t) = \sum_{n=0}^{\infty} \frac{n t^{n-1} H_n(x)}{n!}$$

$$2x \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!} - 2t \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!} = \sum_{n=0}^{\infty} \frac{n t^{n-1} H_n(x)}{n!}$$

Comparing the coefficients of t^n on both sides

$$2x \frac{H_n(x)}{n!} - \frac{2H_{n-1}(x)}{(n-1)!} = \frac{(n+1)H_{n+1}(x)}{(n+1)!}$$

$$2x \frac{H_n(x)}{n!} - \frac{2H_{n-1}(x)}{(n-1)!}\frac{n}{n} = \frac{(n+1)H_{n+1}(x)}{(n+1)n!}$$

$$2x\ H_n(x) = 2n\ H_{n-1}(x) + H_{n+1}(x)$$

3) STATEMENT: $H_n'(x) = 2x\ H_n(x) - H_{n+1}(x)$

Proof: Considering Recurrence relation 1.

$$H_n = 2n\ H_{n-1}(x)$$

Considering Recurrence relation 2

$$2x\ H_n(x) = 2n\ H_{n-1}(x) + H_{n+1}(x)$$

Subtracting eq 2 from eq 1

$$H_n'(x) - 2x\ H_n(x) = 2n\ H_{n-1}(x) - 2n\ H_{n-1}(x) - H_{n+1}(x)$$

$$H_n'(x) - 2x\ H_n(x) = H_{n+1}(x)$$

$$H_n'(x) = 2x\ H_n(x) + H_{n+1}(x)$$

5.7 RODRIGUES FORMULA:

Statement: $H_n(x) = (-1)^n e^{x^2} \frac{d^n(e^{-x^2})}{dx^n}$

Proof: From the generating function of Hermite Polynomial

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!}$$

$$e^{x^2-(t-x)^2} = H_0 + \frac{tH_1(x)}{1!} + \frac{t^2 H_2(x)}{2!} + \ldots\ldots + \frac{t^{n-1} H_{n-1}(x)}{(n-1)!} + \frac{t^n H_n(x)}{(n)!} + \frac{t^{n+1} H_{n+1}(x)}{(n+1)!}$$

Now partially differentiate above equation up to n times with respect to t and then putting t=0

$$e^{x^2}\frac{d^n(e^{-(t-x)^2})}{dt^n}\Big|_{t=0} = \frac{n!\,H_n(x)}{n!}$$

$$e^{x^2}\frac{d^n(e^{-(t-x)^2})}{2t^n}\Big|_{t=0} = H_n(x)$$

t-x = u

dt=du at t=0

u =-x

$$H_n(x) = e^{x^2}\frac{d^n(e^{-u^2})}{du^n}\Big|_{t=0}$$

$$= e^{x^2}\frac{d^n(e^{-x^2})}{d(-x)^n}\Big|_{t=0}$$

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n(e^{-x^2})}{dx^n}$$

$$H_n(-x) = (-1)^n H_n(x)$$

From the generating function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty}\frac{t^n\,H_n(x)}{n!} \qquad \text{------------}(1)$$

Substitute x = -x on both sides

$$e^{-2tx-t^2} = \sum_{n=0}^{\infty}\frac{t^n\,H_n(-x)}{n!} \qquad \text{----------}(2)$$

Substitute t = -t on both sides of (1)

$$e^{-2tx-t^2} = \sum_{n=0}^{\infty}\frac{(-t)^n\,H_n(x)}{n!} \qquad \text{-----------}(3)$$

From (2) and (3)

$$\sum_{n=0}^{\infty}\frac{t^n\,H_n(-x)}{n!} = \sum_{n=0}^{\infty}\frac{(-t)^n\,H_n(x)}{n!}$$

Comparing the coefficients of t^n on both sides

$$\frac{H_n(-x)}{n!} = \frac{(-1)^n H_n(x)}{n!}$$

$$H_n(-x) = (-1)^n H_n(x)$$

1) $H_n' = 2n H_{n-1}(x)$

5.8 PHYSICAL APPLICATIONS:

The **Hermite differential equation** is another key second-order linear differential equation that arises in various physical contexts. Its general form is:

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

where n is a non-negative integer, and y(x)y is the unknown function. The solutions to this equation are known as **Hermite polynomials** for integer n, and they play an important role in various fields of physics. Here are some of the main **physical applications** of the Hermite differential equation:

1) Quantum Mechanics (Harmonic Oscillator)

- **Quantum Harmonic Oscillator**: One of the most important applications of Hermite polynomials is in the solution of the **quantum harmonic oscillator** problem. In quantum mechanics, the Schrödinger equation for a particle in a harmonic potential leads to solutions that are expressed in terms of Hermite polynomials. The energy eigenfunctions of the quantum harmonic oscillator involve Hermite polynomials and a Gaussian factor.

  The radial part of the wavefunction for the harmonic oscillator in one dimension is of the form:

$$\Psi_n(x) = N_n e^{\frac{-x^2}{2}} H_n(x)$$

  Where $H_n(x)$ is the Hermite polynomial of degree n, and N_n is a normalization factor. These wavefunctions describe the allowed quantum states for a particle in a harmonic potential.

2) Optics (Gaussian Beams):

- **Laser Beam Propagation**: Hermite polynomials also appear in the description of **Gaussian beams**, which are solutions to the wave equation in optics. A Gaussian beam's spatial profile can be expressed using Hermite-Gaussian modes, which are the eigenmodes of the paraxial wave equation.

  These modes are of the form:

$$u_{m,n}(x,y,z) = H_m \frac{x}{\text{w}(z)} H_n \frac{y}{\text{w}(z)} \exp - \frac{x^2+y^2}{\text{w}^2(z)}$$

  where $H_m(x)$and $H_n(y)$are Hermite polynomials, and$\text{w}(z)$ is the beam waist that depends on the axial position z. These beams are used in optical systems, such as laser cavities, optical tweezers, and micromanipulation.

3) Heat Conduction Problems (Fourier Series Expansion):

- **Heat Conduction in a Semi-Infinite Solid**: In the study of heat conduction, the **Hermite differential equation** arises in the solution to problems involving the diffusion of heat in a semi-infinite solid. When solving the heat equation using methods such as separation of variables, one obtains solutions that involve Hermite polynomials, especially when the temperature profile exhibits Gaussian behavior.

  The solutions often take the form of **Fourier series** expansions in terms of Hermite functions, which help describe the distribution of heat over time.

4) **Statistical Mechanics (Gaussian Distribution):**

- **Gaussian Distribution and Central Limit Theorem**: In statistical mechanics, the **Gaussian distribution** (which is related to the normal distribution) is closely linked to Hermite polynomials. The probability distribution of particles in certain systems, such as in the case of the velocity distribution of gas molecules in classical thermodynamics, involves a Gaussian function. Hermite polynomials are used in the **expansion** of such distributions in series, particularly in statistical mechanics.

- In the context of the **Central Limit Theorem**, Hermite polynomials appear in the series expansion of the characteristic function of the sum of independent random variables, showing how distributions approach a Gaussian shape as the number of variables increases.

5) Electromagnetic Wave Propagation

- **Waveguides and Fiber Optics**: In the study of wave propagation in waveguides and optical fibers, the **mode solutions** often involve Hermite polynomials. For example, in the analysis of the electromagnetic fields in rectangular waveguides, the field solutions can be expressed as products of Hermite polynomials and sinusoidal functions. This is particularly true for **rectangular waveguides** or **fiber optics**, where the transverse electric and magnetic field components are described by Hermite functions.

6) Vibrations of a Membrane (Membrane Modes)

- **Vibrating Membranes**: In certain problems in acoustics and mechanical vibrations, the modes of vibration of a **two-dimensional membrane** with specific boundary conditions can be described by solutions to the Hermite differential equation. This occurs especially when the boundary conditions are related to Cartesian or polar coordinates, leading to **separation of variables** and solutions that involve Hermite polynomials.

7) Astrophysics (Stellar Structure)

- **Stellar Models**: In some models of **stellar structure**, Hermite polynomials are used in the solution of **stellar equilibrium equations**. For example, certain approximations in the **radiative transfer** equations for stars may involve solutions that include Hermite polynomials, especially when approximating the behavior of radiation in stellar interiors.

8) Acoustics (Vibrations in Cylindrical or Spherical Domains)

- **Acoustic Wave Equations**: In problems involving the propagation of sound or waves in cylindrical or spherical domains (e.g., sound propagation in a cylindrical tube or in spherical cavities), the solutions can involve **Hermite polynomials**. These solutions appear when solving the **acoustic wave equation** in such geometries, especially under conditions that lead to Gaussian-like profiles.

5.9 SUMMARY:

The **Hermite differential equation** is fundamental in various physical fields, with **Hermite polynomials** appearing in numerous contexts, such as:

- **Quantum mechanics** (Harmonic Oscillator Wavefunctions),
- **Optics** (Gaussian Beams and Hermite-Gaussian Modes),
- **Heat conduction** (Fourier Expansions),
- **Statistical mechanics** (Gaussian Distributions),

- **Electromagnetic wave propagation** (Waveguides and Optical Fibers),
- **Acoustics** (Membrane Vibrations),
- **Astrophysics** (Stellar Models).

These applications demonstrate the wide-reaching significance of Hermite polynomials in describing physical phenomena that involve Gaussian-like behavior, radial symmetry, or quantum states.

The **Hermite differential equation** is a second-order linear differential equation of the form:

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

where n is a non-negative integer, and y(x)is the unknown function. The solutions to this equation are the **Hermite polynomials**$H_n(x)$ which are important in many areas of physics and mathematics.

5.10 KEY FEATURES:

1) **General Solution**:
   - For integer n, the solutions are **Hermite polynomials**$H_n(x)$, which are polynomials of degree n.
   - For non-integer n, the solutions involve **generalized Hermite functions**, often expressed as power series.
2) **Orthogonality**:
   - Hermite polynomials are orthogonal with respect to the weight function e^{-x}over the entire real line. This orthogonality property makes them useful in solving problems that involve Gaussian distributions, such as in quantum mechanics and statistical mechanics.
3) **Recurrence Relations**:
   - Hermite polynomials satisfy a recurrence relation, which allows for the construction of higher-order polynomials from lower-order ones
   - $H_{(n+1)}(x) = 2xH_n(x) - 2nH_{(n-1)}(x)$
4) **Asymptotic Behavior**:
   - For large xxx, Hermite polynomials grow rapidly. Their behavior is important in the approximation of functions, especially in the context of asymptotic expansions.

5.11 SELF-ASSESSMENT QUESTIONS:

1) Explain about Hermite polynomials?
2) Briefly explain about Recurrence relations?
3) Explain about Rodrigues formula?

5.12 SUGGESTED READINGS:

Here are some recommended reference books for studying the **Hermite differential equation** and its applications in various fields of physics and mathematics:

1) "Mathematical Methods for Physicists" by George B. Arfken and Hans J. Weber

- This comprehensive book provides a solid treatment of special functions, including the Hermite differential equation. It covers the theory of Hermite polynomials, their properties, and their applications in physics, especially in quantum mechanics and optics.

2) "Mathematics for Physicists" by Peter S. McGrath

- McGrath's book offers a practical approach to solving differential equations encountered in physics, including Hermite differential equations. It also discusses their physical significance and how they relate to quantum mechanics and statistical mechanics.

3) "Methods of Mathematical Physics" by Richard Courant and David Hilbert

- A classic reference on mathematical methods, this two-volume work covers a wide range of topics, including special functions like Hermite polynomials. It provides rigorous mathematical treatments and physical interpretations of these solutions in various contexts.

Prof. Ch. Linga Raju

LESSON-6

LAGUERRE DIFFERENTIAL EQUATION

6.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Laguerre differential equation. The chapter began with understanding of The Power series Solutions, Generating Functions, Rodrigue's formula, Recurrence Relations, Orthogonal Properties Physical Application. After completing this chapter, the student will understand the complete idea about Laguerre Differential Equation.

STRUCTURE:

6.1 INTRODUCTION:

There are very many particular differential equations which find all important place in the scientific applications. Laguerre's second order differential equation with variable coefficients is one such. Particular attention may be drawn to the radial wave equation in quantum mechanics isomorphous with the differential equation in mathematics whose solutions are associated Laguerre functions. But stress is given in this lesson to only Laguerre Polynomials.

6.2 SOLUTION OF LAGUERRE'S DIFFERENTIAL EQUATION:

Laguerre's differential equation may be written as

$xy'' + (1 - x)\, y' + \lambda y = 0$, where λ = constant ------------ (1)

This equation has a singularity at x = 0. But the singularity is non-essential or removable and hence the method of series integration is allowed by Fusch's theorem for solving this equation. For this purpose, we take$y = \sum_{l=0}^{\infty} a_l x^{K+l}$ (where k is constant and $a_0 \neq 0$) as the solution of given differential equation.

Thus $y' = \sum_l (k + l)\, a_l\, x^{k+l-1}$ and $y'' = \sum_l (k + l)(k + l - 1)\, a_l\, x^{k+l-2}$

Substituting these values in equation (1), we have

$$\sum_l (k + l)^2 a_l x^{k+l-1} - \sum_l a_1 (k + l - \lambda)\, x^{k+l} \equiv 0 \quad \text{---------} \quad (2)$$

This equation is true for all the values of x and hence the coefficients of all the powers of x are identically zero. As such equating to zero the coefficient of the lowest power of x, i.e., of x^{k-1}, we have the indicial equation as

$$k^2 a_0 = 0 \qquad \text{----------} \ (3)$$

since $a_0 \neq 0$**;** so equation (2) holds good only if k = 0. Then, we have

$$\sum l^2 a_l x^{l-1} - \sum a_l (l - \lambda)\, x^l = 0 \qquad \text{---------} \ (4)$$

Equating the coefficients of x^j to zero, we have $a_{j+1} = \frac{j - \lambda}{(j+1)^2} a_j$.

This is the recurrence relation for the coefficients.

Thus $\quad a_l = -\lambda a_0 = (-1)\lambda\, a_0$,

$$a_2 = \frac{1-\lambda}{2^2}(-\lambda a_0) = \frac{\lambda(\lambda-1)}{2^2} a_0 = \frac{\lambda(\lambda-1)}{(2!)^2} a_0 = (-1)^2 \frac{\lambda(\lambda-1)}{(2!)^2} a_0.$$

$$a_3 = \frac{2-\lambda}{3^2} a_0 = \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2} a_0 = (-1)^3 \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2} a_0,$$

… … … … … … …

$$a_r = \frac{r - 1 - \lambda}{r^2} a_{r-1} = (-1)^r \frac{\lambda(\lambda-1)\ldots(\lambda - r + 1)}{(r!)^2} a_0.$$

So $y = \sum_{l=0}^{\infty} a_l x^l = a_0\left[1-\lambda x+\frac{\lambda(\lambda-1)}{(2!)^2}x^2-\cdots+(-1)^r\frac{\lambda(\lambda-1)...(\lambda-r+1)}{(r!)^2}x^r+...\right]$------- (5)

If $\lambda = n$, a positive integer, and if, we put $a_0 = n\,!$, (some authors may take $a_0 = 1$) then the solution for y contains only (n + 1) terms and becomes the Laguerre Polynomial of degree n.

Thus (5) becomes $L_n(x) = \sum_{r=0}^{n} \frac{(-1)^2!(\underline{|n})^2}{\underline{|n-r}(\underline{|r})^2} x^r$

$$= (-1)^2\left[x^n - \frac{n^2}{1!}x^{n-1} + \frac{n^2(n-1)^2}{2!}x^{n-2} + \cdots + (-1)^n n!\right] \quad \text{----------} \quad (6)$$

This is the expression for **Laguerre's Polynomial.**

Equation (6) gives $L_n(0) = n\,!$, $L_0(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = x^2 - 4x + 2$,

$L_3(x) = x^3 + 9x^2 - 36x + 6$, $L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 48$, and so on.

Thus a Laguerre's polynomial is the solution of equation

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0 \quad \text{----------} \quad (7)$$

6.3 GENERATING FUNCTION OF LAGUERRE POLYNOMIAL:

$\frac{1}{1-t}e^{-tx}/_{1-t}=\sum_{n=0}^{\infty} t^n L_n(x)$

Solution:

Let $\frac{1}{1-t}e^{\frac{-tx}{1-t}} = \left(\frac{1}{1-t}\right)[1 + \left\{\frac{\frac{-tx}{1-t}}{1!}\right\} + \frac{\{\frac{-tx}{1-t}\}^2}{2!} + \ldots\ldots\ldots \frac{\{\frac{-tx}{1-t}\}^r}{r!} + \ldots\ldots\ldots]$

$$=\left(\frac{1}{1-t}\right)\sum_{r=0}^{\infty}\frac{(-1)^r(\frac{tx}{1-t})^r}{r!}$$

$$=\left(\frac{1}{1-t}\right)\sum_{r=0}^{\infty}\frac{(-1)^r t^r x^r}{r!(1-t)^r} = \sum_{r=0}^{\infty}\frac{(-1)^r x^r t^r}{r!(1-t)^{r+1}}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r x^r t^r}{r!}(1-t)^{-(r+1)}$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots\ldots\ldots$$

$$\frac{1}{1\_t}e^{\frac{-tx}{1-t}} = \sum_{r=0}^{\infty}\frac{(-1)^r x^r t^r}{r!}[1 + t(r+1) + \frac{(r+1)(r+2)}{2!}t^2 + \cdots$$

$$+ \frac{(r+1)(r+2)\ldots.(r+s)}{s!}t^s + \cdots]$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r x^r t^r}{r!}\sum_{r=0}^{\infty}\frac{(r+s)!}{r!s!}t^s$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r x^r (r+s)!}{(r!)^2 s!}t^{r+s}$$

Now by fixing the value s=n-r. Then the coefficient of t^n

$$= \sum_{r=0}^{\infty}\frac{(-1)^r x^r n!}{(r!)^2(n-r)!}$$

s $\geq$0

n – r $\geq$0

n$\geq$0

r$\leq$n

$$\frac{1}{1-t}e^{\frac{-tx}{1-t}} = \left(\sum_{n=0}^{\infty}\frac{(-1)^r x^r n!}{(r!)^2(n-r)!}\right)$$

$$\frac{1}{1-t}e^{\frac{-tx}{1-t}} = \left(\sum_{n=0}^{\infty}t^n L_n(x)\right)$$

6.4 RODRIGUE FORMULA FOR LAGUERRE DIFFERENTIAL EQUATION:

Differentiating generating function equation (18) n times w.r.t.ρ, we have

$$e^x \frac{\partial^n}{\partial\rho^n}\left[(1-\rho)^{-1}e^{-x/(1-\rho)}\right] = L_n(x) + L_{n+1}(x)\,\rho + \ldots. \qquad \text{----------} (22)$$

since all terms up to the term containing ρ^{n-1} vanish when differentiated n times.

But $$\frac{\partial}{\partial\rho}\left[(1-\rho)^{-1}e^{-x/(1-\rho)}\right] = \frac{1-x-\rho}{(1-\rho)^3}\ e^{-x/(1-\rho)}$$

So $$\lim_{\rho\to 0}\frac{\partial}{\partial\rho}\left[(1-\rho)^{-1}e^{-x/(1-\rho)}\right] = (1-x)e^{-x} = \frac{d}{dx}\ (x\ e^{-x}).$$

Similarly, $$\lim_{\rho\to 0}\frac{\partial^2}{\partial\rho^2}\left[(1-\rho)^{-1}e^{-x/(1-\rho)}\right] = \frac{d^2}{dx^2}(x^2\ e^{-x})$$ and so on.

Thus finally, we have $$\lim_{\rho\to 0}\frac{\partial^n}{\partial\rho^n}\left[(1-\rho)^{-1}e^{-x/(1-\rho)}\right] = \frac{d^n}{dx^n}(x^n\ e^{-x})$$

And hence equation (22) for $\rho\to 0$ gives

$$L_n(x) = e^x\frac{d^n}{dx^n}(x^n\ e^{-x}) \qquad \text{----------}\ (23)$$

Which is Rodrigue's representation of Laguerre's polynomial.

The Rodrigue's**;** representation of associated Laguerre's polynomial is given by

$$L_n^k(x) = e^x\ x^{-k}\frac{d^n}{dx^n}(\ e^{-x}x^{n+k}) \qquad \text{----------}\ (24)$$

6.5 RECURRENCE RELATIONS:

1) $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n_1}(x)$

2) $xL'_n(x) = nL_n(x) - nL_{n_1}(x)$

3) $L'_n(x) = -\sum_{r=0}^{n-1} L_r\ (x)$

First recurrence relation:

$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n_{-1}}(x)$

From the generating function of Laguerre polynomial

$$\frac{1}{1-t}e^{\frac{-tx}{1-t}} = \sum_{n=0}^{\infty} t^n L_n(x)$$

Differentiate with respect to'‘t’ on both sides

$$\sum_{n=0}^{\infty} nt^{n-1}L_n(x) = \frac{1}{1-t}e^{\frac{-tx}{1-t}}\left[\frac{(1-t)-t(-1)}{(1-t)^2}\right] - x + e^{\frac{-tx}{1-t}}\frac{1}{(1-t)^2}$$

$$=\frac{1}{1-t}e^{\frac{-tx}{1-t}}\frac{-x}{(1-t)^2} + \frac{-1}{(1-t)^2} + e^{\frac{-tx}{1-t}}$$

$$(1-t)^2\sum_{n=0}^{\infty} nt^{n-1}L_n(x)$$

$$= -\mathrm{x}\sum_{n=0}^{\infty} t^n L_n(x) + (-t)\sum_{n=0}^{\infty} t^n L_n(x)\sum_{n=0}^{\infty} nt^{n-1}L_n(x) + \mathrm{t}^2\sum_{n=0}^{\infty} t^n L_n(x)$$

$$-2t\sum_{n=0}^{\infty} nt^{n-1}L_n(x)$$

$$=-x\sum_{n=0}^{\infty} t^n L_n(x) + \sum_{n=0}^{\infty} t^n L_n(x) - \sum_{n=0}^{\infty} nt^{n+1}L_n(x)$$

Comparing the coefficient of t^n on both sides

$$(n+1)L_{n+1}(x) + (n-1)L_{n_{-1}}(x) - 2nL_n(x) = -xL_n(x) + L_n(x) - L_{n_{-1}}(x)$$

$$(n+1)L_{n+1}(x) = (2n+1)L_n(x) - xL_n(x) - nL_{n_{-1}}(x)$$

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n_{-1}}(x)$$

Second recurrence relation:

$$xL_n'(x) = nL_n(x) - nL_{n_1}(x)$$

From the generating function of Laguerre polynomial

$$\frac{1}{1-t}e^{\frac{-tx}{1-t}} = \sum_{n=0}^{\infty} t^n L_n(x)$$

Derivative with respect to 'x' on both sides

$$\frac{1}{1-t}e^{\frac{-tx}{1-t}}\left(\frac{-t}{1-t}\right) = \sum_{n=0}^{\infty} t^n L_n'(x)$$

$$\frac{-t}{(1-t)}\sum_{n=0}^{\infty} t^n L_n(x) = \sum_{n=0}^{\infty} t^n L_n'(x)$$

$$(1-t)\sum_{n=0}^{\infty} t^n L_n'(x) = -\mathrm{t}\sum_{n=0}^{\infty} t^n L_n(x)$$

$$\sum_{n=0}^{\infty} t^n L_n'(x) - \sum_{n=0}^{\infty} t^{n+1} L_n'(x) = -\sum_{n=0}^{\infty} t^{n+1} L_n(x)$$

Comparing the coefficient t^n on both sides

$$L_n'(x) - L_{n-1}'(x) = -L_{n-1}(x)$$

$$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x) \qquad \ldots\ldots\ldots..(1)$$

Replace 'n' by n+1 in equation 1 then

$$L_{n+1}'(x) = L_n'(x) - L_n(x) \qquad \ldots\ldots\ldots..(2)$$

Consider recurrence relation 1

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n_{-1}}(x) \qquad \ldots\ldots\ldots(3)$$

Derivative above equation with respect to 'x'

$$(n+1)L_{n+1}'(x) = (2n+1-x)L_n'(x) - xL_n'(x) - L_n(x) - nL_{n-1}'(x)$$

Sub $L_n'(x)$ and $L_{n+1}'(x)$ in above equation then

$$(n+1)\{L_n'(x) - L_n(x)\} = (2n+1)\{L_{n-1}'(x) - L_{n-1}(x)\} - (x)L_n'(x) - L_n(x)nL_{n-1}'(x)$$

$$nL'_n(x) + L'_n(x) - nL_n(x) - L_n(x)$$

$$= 2nL'_{n-1}(x) + L'_{n-1}(x) - 2nL_{n-1}(x) - L_{n-1}(x) - xL'_n(x) - L_n(x)$$

$$- nL'_{n-1}(x)$$

$$xL'_n(x) = nL_n(x) - nL_{n_1}(x)$$

Third recurrence relation:

$$L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)$$

From the generating function

$$\frac{1}{1-t} e^{\frac{-tx}{1-t}} = \sum_{n=0}^{\infty} t^n L_n(x)$$

Differentiate with respect to 'x'

$$\sum_{n=0}^{\infty} t^n L'_n(x) = \frac{1}{1-t} e^{\frac{-tx}{1-t}} \left(\frac{-t}{1-t}\right)$$

$$= \left(\frac{-t}{1-t}\right) \sum_{r=0}^{\infty} t^r L_r(x)$$

$$= -\sum_{r=0}^{\infty} t^{r+1} L_r(x)\,(1-\mathrm{t})^{-1}$$

$$= -\sum_{r=0}^{\infty} t^{r+1} L_r(x)\{1 + t + t^2 + \cdots + t^s + \cdots\}$$

$$= -\sum_{r=0}^{\infty} t^{r+1} L_r(x) \sum_{s=0}^{\infty} t^s$$

$$= -\sum_{r,s=0}^{\infty} t^{r+1+s} L_r(x)$$

$$\sum_{n=0}^{\infty} t^n L'_n(x) = -\sum_{r,s=0}^{\infty} t^{r+1+s} L_r(x)$$

Let r+s+1=n

S=n-r-1

s≥=0

n-r-1≥ =0

n-1 ≥ r

$$\sum_{n=0}^{\infty} t^n L_n'(x) = -\sum_{r=0}^{n-1} t^n L_r(x)$$

Comparing the coefficient of t^n on both sides

$$L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)$$

6.6 ORTHOGONAL PROPERTIES OF LAGUERRE POLYNOMIALS:

Laguerre's differential equation is not self*adjoint and thus Laguerre's polynomials $L_n(x)$ do not by themselves form an orthogonal set.

However, the related set of functions

$$\phi_n(x) = \frac{1}{n!} e^{-x/2} L_n(x) \qquad ---------- (25)$$

where $e^{-x/2}$ is the weight function of $L_n(x)$ is orthogonal for the interval $0 \le x \le \infty$, i.e.,

$$\int_0^\infty e^{-x} \frac{L_m(x)}{m!} \frac{L_n(x)}{n!} dx = \int_0^\infty \phi_m(x)\, \phi_n(x)\, dx = \delta_{m,n} \qquad ---------- (26)$$

It can be proved as follows:

We know that $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$. Multiplying both sides with $e^{-x}x^m$ and integrating w.r.t. x between the limits 0 to ∞, we get

$$\int_0^\infty e^{-x} x^m L_n(x)\, dx = \int_0^\infty x^m \frac{d^n}{dx^n}(x^n e^{-x}) dx$$

$$=\left[x^m \frac{d^{n-1}}{dx^{n-1}} x^n e^{-x}\right]_0^\infty - \int_0^\infty mx^{m-1} \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x})dx = (-1)m\int_0^\infty x^{m-1} \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x})dx$$

$$= (-1)^2 m(m-1) \int_0^\infty x^{m-2} \frac{d^{n-2}}{dx^{n-2}} (x^n e^{-x})dx = \ldots\ldots\ldots..$$

$$= (-1)^2 m \int_0^\infty \frac{d^{n-m}}{dx^{n-m}} (x^n e^{-x})dx \qquad \text{(on integrating by parts)}$$

$= 0$ if $n > m$.

similarly, $\int_0^\infty e^{-x} x^m L_m(x)\, dx = 0$ if $m > n$.

But $L_n(x)$ is a polynomial of degree n in x and $L_m(x)$ is a polynomial of degree m in x.

Therefore, $\int_{-\infty}^\infty e^{-x} L_m(x)\, L_n(x)\, dx = 0$ for $m > n$ and for $m < n$

Or $$\int_{-\infty}^\infty e^{-x} \frac{L_m(x)}{m!} \frac{L_n(x)}{n!} dx = 0 \text{ if } m \neq 0. \quad \text{---------- (27)}$$

For $m = n$, $\int_{-\infty}^\infty e^{-x} \{L_n(x)\}^2 dx = (-1)^n \int_{-\infty}^\infty e^{-x} x^n L_n(x)\, dx$

(since the term of degree n in $L_n(x)$ is $(-1)^n x^n$).

Thus $$\int_0^\infty e^{-x} \{L_n(x)\}^2 dx = (-1)^n \int_0^\infty e^{-x} x^n e^x \frac{d^n}{dx^n} (x^n e^{-x})dx$$

$$= (-1)^n n! \int_0^\infty x^n (-1)^n e^{-x} dx = n! \int_0^\infty x^n e^{-x} dx = (n!)^2 \quad \text{--------- (28) or}$$

$$\int_0^\infty \frac{1}{n!} e^{-x/2} L_n(x) \cdot \frac{e^{-x/2}}{n!} L_n(x)\, dx = 1$$

Thus from equations (27) and (28), we get

$$\int_0^\infty e^{-x/2} \frac{L_m(x)}{m!} e^{-x/2} \frac{L_n(x)}{n!} dx = \delta_{m,n}$$

$\int_0^\infty \phi_m(x)\, \phi_n(x)\, dx = \delta_{m,n}$

Second method: To prove that $\int_0^\infty e^{-x} \frac{L_m(x)}{\underline{|m}} \cdot \frac{L_n(x)}{\underline{|n}}\, dx = \delta_{m n}$

Proof: We have $\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{\underline{|n}} = \frac{1}{1-t}\, e^{\frac{-tx}{1-t}}$

and $\sum_{m=0}^{\infty} s^m \frac{L_m(x)}{\underline{|m}} = \frac{1}{1-s}\, e^{\frac{-sx}{1-s}}$

Further,

$$\sum_{m,n=0}^{\infty} e^{-x} t^n s^m \frac{L_n(x)}{\underline{|n}} \frac{L_m(x)}{\underline{|m}} = e^{-x} \frac{1}{1-t} \frac{1}{1-s}\, e^{\frac{-tx}{1-t}} \cdot e^{\frac{-sx}{1-s}}$$

Integrating both sides w.r.t. x between the limits 0 to ∞, we can have a typical integral

$\int_0^\infty e^{-x} \frac{L_n(x)}{\underline{|n}} \cdot \frac{L_m(x)}{\underline{|m}}\, dx$ = coefficient of $t^n s^m$ in the expansion of

$$\int_0^\infty e^{-x} \frac{1}{(1-t)(1-s)}\, e^{\frac{-tx}{1-t}} \cdot e^{\frac{-sx}{1-s}} \cdot dx$$

But $\int_0^\infty e^{-x} \frac{1}{(1-t)(1-s)}\, e^{\frac{-tx}{1-t}} \cdot e^{\frac{-sx}{1-s}} = \frac{1}{(1-t)(1-s)} \int_0^\infty e^{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]} \cdot dx$

$$= \frac{1}{(1-t)(1-s)\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]} \cdot \left[e^{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]} \right]_{\lambda=0}^{\infty}$$

$$= \frac{1}{-1+ts}(0-1) = \frac{1}{1+ts} = [1 + ts + (ts)^2 + \ldots\ldots]$$

Here the coefficient of $t^n s^m$ is zero ($m \neq n$) and 1 for $m = n$.

$$\therefore \int_0^\infty e^{-x} \frac{L_n(x)}{\underline{|n}} \cdot \frac{L_m(x)}{\underline{|m}} \, dx = \delta_{m\,n}$$

6.7 PHYSICAL APPLICATIONS:

Laguerre differential equations have various applications in physics, particularly in areas involving quantum mechanics, optics, and wave phenomena. Some of the most prominent physical applications are:

1. Quantum Mechanics:

- **Radial Wavefunction in Hydrogen-like Atoms:** In the Schrödinger equation for the hydrogen atom (or hydrogen-like atoms), the radial part of the wavefunction satisfies the Laguerre differential equation. When solving for the energy eigenfunctions, we often encounter associated Laguerre polynomials. These polynomials appear in the radial wavefunctions Rnl(r)R_{nl}(r)Rnl(r) for the hydrogen atom and other systems with central potentials.
- The general form of the radial part of the wavefunction is derived from the Laguerre differential equation and is written in terms of associated Laguerre polynomials Ln(l)(x)L_n^{(l)}(x)Ln(l)(x). These polynomials represent the quantum states of an electron in an atom, especially for discrete energy levels.

2. Optics:

- **Laguerre-Gaussian Beams:** Laguerre polynomials appear in the description of laser beams with vortex-like properties, known as Laguerre-Gaussian beams. These beams are solutions to the Helmholtz equation and are used in optical systems that manipulate light in terms of its angular momentum. The intensity distribution of such beams can be expressed in terms of Laguerre polynomials, and they have applications in optical trapping, micromanipulation, and quantum optics.

3. Radial Vibrations and Waveguides:

- **Vibrations of Circular Membranes:** For circular membranes (such as those in drumheads or membranes in acoustics), the solutions to the wave equation can involve Laguerre functions. The modes of vibration for such membranes can be described by functions that include associated Laguerre polynomials in the radial direction, especially in systems with circular symmetry.
- **Waveguides and Optical Fibers:** In waveguide theory, particularly in the context of optical fibers, the radial modes of light propagation can be described by the Laguerre differential equation. Solutions often involve Laguerre polynomials that describe the spatial variation of the modes in the radial direction.

4. Hydrodynamics:

- **Flow Problems in Cylindrical Coordinates:** In fluid dynamics, certain cylindrical flow problems, such as those involving vortex motion or flow in cylindrical pipes, can lead to differential equations that involve Laguerre functions. These functions help in solving flow problems with specific boundary conditions, such as those found in fluid-filled tubes or rotating systems.

5. Electrostatics:

- **Potential Theory:** In electrostatics, solutions to Laplace's equation in cylindrical coordinates (with specific boundary conditions) can involve Laguerre polynomials. These solutions are useful for systems with cylindrical symmetry, such as in the analysis of electric fields within coaxial cables or cylindrical conductors.

6. Nuclear Physics:

- **Wave functions for Nuclear Potentials:** In the study of nuclear potentials, the radial solutions to the Schrödinger equation often involve Laguerre polynomials. These solutions describe the behavior of nucleons within a nucleus, especially when approximating the behavior of particles under certain potential models.

These applications highlight the versatility of Laguerre differential equations in physical problems that exhibit cylindrical symmetry or involve systems with quantized states, such as in quantum mechanics and wave theory.

6.8 SUMMARY:

The **Laguerre differential equation** is a second-order linear differential equation of the form:

$$\frac{d^2y}{dx^2}\,x + (b - x)\frac{dy}{dx} + ay = 0$$

where a and b are constants, and y(x) is the unknown function. It is a special case of the general class of **generalized Laguerre equations** and has important mathematical and physical applications.

6.9 KEY TERMS:

1) **Solution Structure**: The solutions to the Laguerre differential equation are given by the **Laguerre polynomials** $L_n^{(a)}(x)$for integer values of nnn, which are a family of orthogonal polynomials. For non-integer values of nnn, the solutions involve **generalized Laguerre functions**.

2) **Orthogonality**: The Laguerre polynomials $L_n^{(a)}(x)$ are orthogonal with respect to the weight function e^{-x} on the interval (0,∞). This orthogonality property is crucial for their use in various physical applications, such as quantum mechanics and optics.

6.10 SELF ASSESSMENT QUESTIONS:

1) Explain about the power series solution of Laguerre differential equations?
2) Explain about Rodrigues formula for laguerre polynomial?
3) Briefly explain about Orthogonal properties of laguerre polynomial?

6.11 SUGGESTED READINGS:

Here are some standard reference books that cover **Laguerre differential equations** and their applications in various fields:

1) **"Mathematical Methods for Physicists" by George B. Arfken and Hans J. Weber**
   - This comprehensive book provides an extensive treatment of various mathematical techniques used in physics, including a detailed section on **special functions** like Laguerre polynomials. It explains their derivation, properties, and applications in quantum mechanics and other areas of physics.
2) **"Mathematics for Physics and Physicists" by Peter S. McGrath**
   - McGrath's book includes sections on solving differential equations commonly encountered in physics, including the **Laguerre differential equation**. It focuses on practical applications and solving techniques, making it useful for physicists.
3) **"Methods of Mathematical Physics" by Richard Courant and David Hilbert**
   - A classical reference for mathematical methods in physics, this two-volume work discusses **special functions** and differential equations in depth. It includes sections on **Laguerre polynomials** and their applications in solving physical problems.

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LESSON-7

INTEGRAL TRANSFORMS

7.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Integral Transform. The chapter began with understanding of Laplace Transforms-definitions-Properties, Derivative of Laplace transform of a derivative, Laplace transform of periodic function, evaluation of Laplace transform. After completing this chapter, the student will understand the complete idea about Integral Transform.

STRUCTURE:

7.1 INTRODUCTION:

The integral transform f(s) of a function F(t) is defined as

$$f(s) = I\{f(t)$$

$$= \int_a^b k(s,t)F(t)dt$$

Where $k(s,t)$ is Kernel Transform and 's' is a parameter real or variable.

Depending upon the type of Kernel transform and range of integration different types of integral transforms are obtained.

1) $k(s,t) = e^{-st}$ then

$f(s) = \int_0^\infty e^{-st} F(t)dt$

This is known as Laplace Transform

2) $k(s,t) = tJ_n(s,t)$ then

$$f(s) = \int_0^\infty tJ_n(s,t)F(t)dt$$

This is known as Hankel Transform (Fourier Bessel's transform)

Here $J_n(s,t)$ is Bessel's function of order x

3) $k(s,t) = t^{s-1}$ then

$f(s) = \int_0^\infty t^{s-1} F(t)dt$

This is known as Mellin transform

4) $k(s,t) = e^{-ist}$ then

$f(s) = \int_0^\infty e^{-ist} F(t)dt$ is known as Fourier transform

5) $k(s,t) = Cosst$

$f(s) = \int_o^\infty CosstF(t)dt$ is known as Fourier Cosine Transform

6) $k(s,t) = Sinst$

$f(s) = \int_o^\infty SinstF(t)dt$ is known as Fourier Sine Transform

7.2 DEFINITION OF LAPLACE TRANSFORM:

If F(t) be a function of t defined by all positive values i.e, t≥0 then the Laplace Transform of F(t) is denoted by L{f(t)} or f(s) is defined by the expression

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st}F(t)dt$$

If the $\int_0^\infty e^{-st}F(t)dt$ converges for some values of s then the $L\{F(t)\}$ is said to exist otherwise it does not exist

7.3 PROPERTIES OF LAPLACE TRANSFORM:

1) Linear property:

If $c_1 and c_2$ are any constants and $f_1(s), f_2(s)$ are Laplace transform of $F_1(t), F_2(t)$ respectively then according to Linear property

$$L\{c_1F_1(t) + c_2F_2(t)\} = c_1L\{F_1(t)\} + c_2L\{F_2(t)\}$$

$$= c_1f_1(s) + c_2f_2(s)$$

Proof:

Since $f_1(s), f_2(s)$ are Laplace transform of $F_1(t), F_2(t)$ respectively then

$$f_1(s) = L\{\{F_1(t)\} = \int_0^\infty e^{-st}F_1(t)dt$$

$$f_2(s) = L\{F_2(t)\} = \int_0^\infty e^{-st}F_2(t)dt$$

L.H.S=$L\{c_1F_1(t)\} + c_2\{F_2(t)\}$

$$= \int_0^\infty e^{-st}\{c_1F_1(t)\} + c_2\{F_2(t)\}dt$$

$$= c_1\int_0^\infty e^{-st}F_1(t)dt + c_2\int_0^\infty e^{-st}F_2(t)dt$$

$$= c_1f_1(s) + c_2f_2(s)$$

=R.H.S

Hence proved

2) Shifting or Translation Property:

a) First shifting property

b) Second shifting property

a) First shifting property:

If f(s) be the Laplace transform of F(t) then

$$L\{e^{at}F(t)\} = f(s-a)$$

Proof:

Since f(s) is Laplace transform of F(t) then

$$f(s) = L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt$$

$$L\{e^{at}F(t)\} = \int_0^\infty e^{-st}e^{at}F(t)dt$$

$$= \int_0^\infty e^{-t(s-a)}(F(t)dt$$

Let (s-a) be p

$$L\{e^{at}F(t)\} = \int_0^\infty e^{-pt}(F(t)dt$$

$$= f(p)$$

$$= f(s-a)$$

$$L\{e^{at}F(t)\} = f(s-a)$$

b) Second shifting property:

If f(s) is Laplace Transform of F(t) and another function G(t) which is defined as

$$G(t) = \begin{cases} F(t-a) & for t > a \\ 0 & for t < a \end{cases}$$ then

$$L\{G(t)\} = e^{-as}f(s)$$

Proof:

Since f(s) is Laplace Transform of F(t)

$$f(s) = L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt$$

$$L\{G(t)\} = \int_0^{\infty} e^{-st} G(t)dt$$

$$L\{G(t)\} = \int_0^{a} e^{-st} G(t)dt + \int_a^{\infty} e^{-st} G(t)dt$$

$$= \int_0^a e^{-st}(0)dt + \int_a^{\infty} e^{-st} F(t-a)dt$$

Let $(t-a) = x$

$t = x + a$

$dt = dx$

$$L\{G(t)\} = \int_0^{\infty} e^{-s(x+a)} F(x)dx$$

$$= e^{-as} \int_0^{\infty} e^{-sx} F(x)dx$$

$$L\{G(t)\} = e^{-as} f(s)$$

3) Change of scalar property:

If f(s) is Laplace transform of F(t) then

$$L\{F(at)\} = \frac{1}{a} f \frac{s}{a}$$

Proof:

Since f(s) is Laplace Transform of F (t)

$$f(s) = L\{F(t)\} = \int_0^{\infty} e^{-st} F(t)dt$$

$$L\{F(t)\} = \int_0^{\infty} e^{-st} F(at)dt$$

Let at $x = dx = \frac{dx}{a}$

$t = \frac{x}{a}$

$$L\{F(at)\} = \int_0^\infty e^{-s(\frac{x}{a})} F(x) \frac{dx}{a}$$

$$L\{F(at)\} = \frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})x} F(x) dx$$

$$L\{F(at)\} = \frac{1}{a} f(\frac{s}{a})$$

Problems:

1. Find the Laplace transform of the following function

$\boldsymbol{i)F(t) = 1 ii)F(t) = t iii)F(t) = t^n}$

We know that

$$f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$$

i)$L\{(1)\} = \int_0^\infty e^{-st} dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$L\{(1)\} = \frac{1}{s}$$

ii) $L\{(t)\} = \int_0^\infty e^{-st} dt$

$$= \left[t\frac{e^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}.1.dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{1}{s}(\frac{1}{s})$$

$$L\{(t)\} = \frac{1}{s^2}$$

iii) $L\{t^n\} = \int_0^\infty e^{-st} t^n dt$

$$= \left[\{t^n\}\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty e^{-st} n t^{n-1} dt$$

$$= \left[\{t^n\}\frac{e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt$$

$$= \frac{n}{s}\left[t^{n-1}\frac{e^{-st}}{-s}\right]_0^\infty + \frac{n-1}{s}\int_0^\infty e^{-st} t^{n-2} dt$$

$$= \frac{n(n-1)}{s^n}\int_0^\infty e^{-st} t^{n-2} dt$$

$$= \frac{n(n-1)(n-2)\ldots\ldots 3.2.1}{s^n}\int_0^\infty e^{-st} t^0 dt$$

$$= \frac{n!}{s^n}\frac{1}{s}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\sqrt{n} + 1}{s^{n+1}}$$

$$L\left\{t^{\frac{1}{2}}\right\} = \frac{\sqrt{(\frac{1}{2} + 1)}}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{s^{\frac{3}{2}}} =$$

2) $L\{e^{at}\}$ =? *find tha laplace transform*

$$f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$$

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-t(s-a)} dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{e^{-at}\} = \frac{1}{s+a}$$

7.4 DERIVATIVE OF LAPLACE TRANSFORM:

If $f(s)$ is Laplace Transform of a function $F(t)$ then

$i) f'(s) = (-1)L\{tF(t)\}$

$ii) f''(s) = (-1)^2 L\{t^2 F(t)\}$

$iii) f'''(s) = (-1)^3 L\{t^3 F(t)\}$

$iv) f^n(s) = (-1)^n L\{t^n F(t)\}$

Proof:

$i)\ since f(s) is a L.T of F(t)$

$$f(s) = \int_0^\infty e^{-st}F(t)dt$$

Differentiate on both sides with respect to ‘s’

$$\frac{d}{ds}\{f(s)\} = \int_0^\infty \frac{d}{ds} e^{-st}F(t)dt$$

$$f(s) = \int_0^\infty e^{-st}(-t)F(t)dt$$

$$= (-1)\int_0^\infty e^{-st}(t)F(t)dt$$

$$f'(s) = (-1)L\{tF(t)\}$$

$$f(s) = \int_0^\infty e^{-st}F(t)dt$$

Differentiate on both sides with respect to ‘s’

$$\frac{d}{ds}\{f(s)\} = \int_0^\infty \frac{d}{ds} e^{-st} F(t) dt$$

$$f(s) = \int_0^\infty e^{-st}(-t)F(t)dt$$

$$= (-1)\int_0^\infty e^{-st}(t)F(t)dt$$

Again differentiate the above equation

$$f''(s) = (-1)\int_0^\infty \frac{d}{ds}(e^{-st})tF(t)dt$$

$$f''(s) = (-1)^2 \int_0^\infty e^{-st}t^2F(t)dt$$

$$f''(s) = (-1)^2 L\{t^2F(t)\} \qquad \ldots\ldots..(3)$$

$$iii)\ f(s) = \int_0^\infty e^{-st}F(t)dt$$

Differentiate equation (3) with respect to‘s’ on both sides

$$\frac{d}{ds}f''(s) = (-1)^2\{\int_0^\infty \frac{d}{ds} e^{-st}\, t^2F(t)\}$$

$$f'''(s) = (-1)^2\{\int_0^\infty e^{-st}\,(-t)t^2F(t)\}$$

$$f'''(s) = (-1)^3\left\{\int_0^\infty e^{-st}\, t^3F(t)\right\}dt$$

$$f'''(s) = (-1)^3 L\{t^3F(t)\}$$

Similarly

$$f^n(s) = (-1)^n L\{t^nF(t)\}$$

7.5 LAPLACE TRANSFORM OF DERIVATIVES:

If $f(s)$ is Laplace Transform of a function $F(t)$ then

$i)\ L\{F'(t)\} = sf(s) - F(0)$

$ii)\ L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

$iii)\ L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$

$iv)\ L\{F^n(t)\} = s^n f(s) - s^{n-1} f(0) \ldots\ldots - sF^{n-2}(0) - F^{n-1}(0)$

Proof:

$i)\ Since f(s)$ is a L$\{F(t)\}$

$$f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

$$L\{F^1(t)\} = \int_0^\infty e^{-st} F'(t)dt$$

$$= [e^{-st} F^1(t)]_0^\infty - \int_0^\infty (-s) e^{-st} F(t)dt$$

$$L\{F^1(t)\} = -F(0) + s\int_0^\infty e^{-st} F(t)dt$$

$$= -F(0) + sf(s)$$

$$L\{F^1(t)\} = sf(s) - F(0)$$

$$ii)\ L\{F''(t)\} = \int_0^\infty e^{-st} F''(t)dt$$

$$= [e^{-st} F^1(t)]_0^\infty - \int_0^\infty (-s) e^{-st} F'(t)dt$$

$$= -F'(0) + s\int_0^\infty e^{-st} F'(t)dt$$

$$= -F'(0) + s(sf(s) - F(0))$$

$$= -F'(0) + s^2 f(s) - sF(0))$$

$$L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

$iii)\ L\{F'''(t)\} = \int_0^\infty e^{-st} F'''(t)dt$

$$= [e^{-st}F^1(t)]_0^\infty - \int_0^\infty (-s)e^{-st}F''(t)dt$$

$$= -F''(0) + s[s^2 f(s) - sF(0) - F'(0))$$

$L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$

Similarly

$$iv)\ L\{F^n(t)\} = s^n f(s) - s^{n-1} f(0) \ldots\ldots - sF^{n-2}(0) - F^{n-1}(0)$$

7.5 LAPLACE TRANSFORM OF PERIODIC FUNCTION:

If $F(t)$ is a periodic function with a period 'T' and $F(t + nT) = F(T) where$

$n = 0,1,2,3 \ldots\ldots$ then

$L\{F(t)\} = \dfrac{1}{1 - e^{-st}} \int_0^T e^{-st} t F(t)dt$

Proof:

Since f(s) is a Laplace Transform of F(t)

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt$$

$$= \int_0^T e^{-st}F(t)dt + \int_T^{2T} e^{-st}F(t)dt + \cdots + \int_{nT}^{(n+1)T} e^{-st}F(t)dt + \cdots.$$

$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st}F(t)dt$

Let *t=x+nT*

dt=dx

$$L\{F(t)\} = L\{F(x + nT)\}$$

$$L\{F(x + nT)\} = \sum_{n=0}^{\infty} \int_{0}^{T} e^{-s(x+nT)} F(x + nT))dx$$

$$L\{F(x)\} = \sum_{n=0}^{\infty} \int_{0}^{T} e^{-s(x+nT)} F(x))dx$$

$$\sum_{n=0}^{\infty} e^{-snt} \int_{0}^{T} e^{-sx} F(x))dx$$

$$L\{F(x)\} = (1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots) \int_{0}^{T} e^{-sx} F(x))dx$$

$$L\{F(x)\} = (1 - e^{-sT})^{-1} \int_{0}^{T} e^{-sx} F(x))dx$$

$L\{F(x)\} = \frac{1}{1-e^{-st}} \int_0^T e^{-sx} tF(x)dx$ for periodic function

$$L\{F(t)\} = \frac{1}{1 - e^{-st}} \int_{0}^{T} e^{-st} tF(t)dt$$

7.7 EVALUATION OF LAPLACE TRANSFORMS:

The **evaluation of Laplace transforms** involves transforming a given time-domain function (typically a function of t, denoted as f(t) into a complex frequency-domain function (denoted as F(s), where s is a complex variable.

The **Laplace transform** of a function f(t) is defined as:

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st} F(t) dt$$

where:

- t is the time variable (typically t≥0),
- s is a complex number s=σ+iω (where σ and ω are real numbers),
- f(t) is the original time-domain function.

Basic Laplace Transforms:

Here are a few standard Laplace transforms for commonly encountered functions:

1) **Constant Function:**

$L(1) = \frac{1}{s},\ for \Re(s) > 0$

2) **Exponential Function:**

$L(e^{at}) = \frac{1}{s-a},\ for \Re(s) > a$

3) **Sine Function:**

$L(\sin(at)) = \frac{1}{s^2+a^2},\ for \Re(s) > 0$

4) **Cosine Function:**

$L(\cos(at)) = \frac{1}{s^2+a^2},\ for \Re(s) > 0$

5) **Power Function (t^n):**

$L(t^n) = \frac{n!}{s^{n+1}} for \Re(s) > 0$

6) **Unit Step Function (Heaviside Function):**

$$L(\mathrm{u}(\mathrm{t}-\mathrm{a})) = \frac{e^{-as}}{s} for \Re(s) > 0$$

where u(t−a) is the Heaviside step function, which is 0 for t<a and 1 for t≥a.

7) **Delta Function (Dirac Delta Function):**

$$L(\delta(t - a)) = e^{-as} \quad \text{for any s}$$

Evaluation Process:

To evaluate the Laplace transform of a function f(t), follow these steps:

1. **Express f(t)** in a form that matches known transforms or can be simplified to match a known form.
2. **Use standard Laplace transform formulas** (like those listed above) to find the Laplace transform of each term.
3. **For more complex functions**, you might need to apply the following techniques:
   - **Linearity**: If $f(t) = f_1(t)+f_2(t) then L\{f(t)\} = L\{f_1(t)\}+L\{f_2(t)\}$
   - **Shifting in time**: If $f(t) = e^{at}g(t), then L\{e^{at}g(t)\} = F(s - a)$
   - **Convolution**: If $f(t) = g(t) * h(t) then L\{f(t)\} = L\{g(t)\}. L\{h(t)\}$
4. **If necessary, simplify the result** in terms of s, and solve for the desired function in the s-domain.

Example: Evaluate the Laplace transform of $f(t) = te^{2t}$

Let's evaluate the Laplace transform of $f(t) = te^{2t}$:

1. **Identify the form**: We know that the Laplace transform of e^{at} is $\frac{1}{s-a}$ and the Laplace transform of t^n is $\frac{n!}{s^{n+1}}$
2. **Use the shifting property**: Since we have both t and e^{2t} we can apply the d=formula for the Laplace Transform of $t. e^{2t}$. This is given by:

$$L\{te^{at}\} = \frac{1}{(s-a)^2}$$

3. **Plug in the value a=2**:

$$L\{te^{2t}\} = \frac{1}{(s-2)^2}$$

Thus, the Laplace transform of $te^{2t} is \frac{1}{(s-2)^2}$.

7.8 SUMMARY:

Integral Transforms are mathematical tools that convert functions from one domain into another, often simplifying the process of solving equations, especially differential equations. They work by integrating the original function with a kernel function, transforming it into a new function in a different domain.

The integral transform f(s) of a function F(t) is defined as

General form of Integral Transform:

$f(s) = I\{f(t)\}$

$= \int_a^b k(s,t)F(t)dt$

Where $k(s,t)$ is Kernel Transform and 's' is a parameter real or variable.

7.9 KEY TERMS:

1) **Kernel Function** – The core function K(s,t) used in the transform.
2) **Transformation** – Converting a function from one domain to another.
3) **Inverse Transform** – Process of converting back to the original domain.
4) **Integral** – Integration is the main operation in integral transforms.
5) **Domain** – The original and transformed domains (e.g., time and frequency).
6) **Convergence** – Conditions under which the integral exists and converges.
7) **Linearity** – Property that $T\{af(t) + bg(t)\} = aF(s) + bG(s)$.
8) **Superposition** – Solution method using the sum of solutions for linear systems.

9) **Differential Equation –** Common application of integral transforms.

10) **Initial Conditions –** Used in transforms like Laplace for solving differential equations.

7.10 SELF ASSESSMENT QUESTIONS:

1) Briefly explain about Laplace Transform of definition and properties?

2) Explain about Laplace Transform of derivatives?

3) Find the Laplace Transform of

   i) sinhat ii) coshat iii) sinat iv) cosat

7.11 SUGGESTED READINGS:

1) "Schaum's Outline of Laplace Transforms" by Murray R. Spiegel

- **Overview**: Offers a concise review of Laplace transforms with solved problems and exercises.
- **Why Read**: Ideal for quick learning and practice, especially for students preparing for exams.

2) "Integral Transforms and Their Applications" by Lokenath Debnath and Dambaru Bhatta

- **Overview**: Comprehensive coverage of integral transforms, including Laplace, Fourier, Mellin, and Hankel transforms. It focuses on applications in engineering, physics, and applied mathematics.
- **Why Read**: Excellent for both theory and practical applications, with plenty of examples and exercises.

3) "The Laplace Transform: Theory and Applications" by Joel L. Schiff

Overview: In-depth exploration of the Laplace transform, including theory, applications, and computational techniques.

- **Why Read**: Focused entirely on the Laplace transform, providing detailed theory and applications in engineering and physics.

Prof. Ch. Linga Raju

LESSON-8

INVERSE LAPLACE TRANSFORMS

8.0 AIMS AND OBJECTIVES:

The primary goal of this chapter is to understand the concept of Inverse Laplace Transform. The chapter began with understanding of Inverse Laplace Transform properties, evaluation of inverse Laplace Transform, elementary function method, Partial fraction method, Solution of ordinary differential equation by using Laplace Transform method. After completing this chapter, the student will understand the complete idea about Inverse Laplace Transform.

STRUCTURE:

8.1 INTRODUCTION:

If $f(s)$ is a Laplace Transform of a function $F(t)$i.e $L\{F(t)\} = f(s)$then $F(t)$ is called an inverse laplace transform of $f(s)$ and is denoted by

$$F(t) = L^{-1}\{f(s)\}$$

Where L^{-1} is called Inverse Laplace Transformation operator

$f(s)L^{-1}\{f(s)\} = F(t)$

$\frac{1}{s}$ 1

$\frac{1}{s^2}$ t

$\frac{1}{s^{n+1}}\frac{t^n}{n!}$

$\frac{1}{s-a}e^{at}$

$\frac{1}{s+a}e^{-at}$

$\frac{1}{(s-a)^{n+1}}e^{at}\frac{t^n}{n!}$

$\frac{1}{s^2+a^2}$ or $\frac{a}{s^2+a^2}$ $Sinat\ or\ \frac{sinat}{a}$

$\frac{s}{s^2+a^2}$ $cosat$

$\frac{a}{s^2-a^2}$ $sinhat$

$\frac{s}{s^2-a^2}$ $coshat$

$\frac{b}{(s-a)^2+b^2}e^{at}sinat$

$\frac{s}{(s-a)^2+b^2}e^{at}cosbt$

8.2 PROPERTIES OF INVERSE LAPLACE TRANSFORM:

1) Linear Property:

If $f_1(s) and\ f_2(s)$ be the Laplace Transform of $F_1(t) and\ F_2(t)$ then

$$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$$

Where $c_1 and\ c_2$ are constants

Proof:

Since $f_1(s) and\ f_2(s)$ are Laplace Transforms of $F_1(t) and\ F_2(t)$ then

$L\{F_1(t)\} = f_1(s)$

$F_1(t) = L^{-1} f_1(s)$

And $L\{F_2(t)\} = f_2(s)$

$F_2(t) = L^{-1} f_2(s)$

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = \int_0^\infty e^{-st}\{c_1 F_1(t) + c_2 F_2(t)\}dt$$

$$= \int_0^\infty e^{-st}\left\{c_1 F_1(t)\}dt + \int_0^\infty e^{-st} c_2 F_2(t)\right\}dt$$

$$= c_1 \int_0^\infty e^{-st} F_1(t)dt + c_2 \int_0^\infty e^{-st} F_2(t)dt$$

$$= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$

$$= c_1 f_1(s) + c_2 f_2(s)$$

$\{c_1 F_1(t) + c_2 F_2(t)\} = L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\}$

$c_1 L^{-1} f_1(s) + c_2 L^{-1} f_2(s) = L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\}$

$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$

2 (i) First Shifting Property:

If $L^{-1}\{f(s)\} = F(t)$ then

$L^{-1}\{f(s-a) = e^{at}F(t)$

Proof:

Since f(s) is the Laplace Transform of a function F(t)

$f(s) = L\{F(t)\}$

$$L\{e^{at}F(t)\} = \int_0^\infty e^{-st}e^{at}F(t)dt$$

$$= \int_0^\infty e^{-t(s-a)}F(t)dt$$

$L\{e^{at}F(t)\} = f(s-a)$

Applying inverse Laplace Transform on both sides

$\{e^{at}F(t)\} = L^{-1}\{f(s-a)\}$

ii) Second shifting property:

If f(s) is Laplace Transform of F(t) then

$L^{-1}\{e^{-as}f(s)\} = G(t)$

Where $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < 0 \end{cases}$

Proof:

Since f(s) is Laplace Transform of F(t) i.e

$L\{F(t)\} = f(s)$

Now $L\{G(t)\} = \int_0^\infty e^{-st}G(t)dt$

$$= \int_0^a e^{-st}G(t)dt + \int_a^\infty e^{-st}G(t)dt$$

$$= 0 + \int_{a}^{\infty} e^{-st} F(t-a) dt$$

Let t-a=x

dt=dx

$$L\{G(t)\} = \int_{a}^{\infty} e^{-s(ta+x)} F(x)\, dx$$

$$= e^{-as} \int_{a}^{\infty} e^{-sx} F(x)\, dx$$

$$= e^{-as} f(s)$$

$$G(t) = L^{-1}\{e^{-as} f(s)\}$$

3) Change of scalar property:

If $L^{-1}\{f(s)\} = F(t)$ then

$$L^{-1}\{f(as)\} = \frac{1}{a} F(\frac{t}{a})$$

Proof:

By the definition of Laplace Transform

$$f(s) = \int_{a}^{\infty} e^{-st} F(t)\, dt$$

$$f(as) = \int_{a}^{\infty} e^{-ast} F(t)\, dt$$

Let $at = x; dt = \frac{dx}{a}$

$$f(as) = \int_{0}^{\infty} e^{-sx} F\left(\frac{x}{a}\right) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-sx} F\left(\frac{x}{a}\right) dx$$

$$= \frac{1}{a} L\left\{F\left(\frac{x}{a}\right)\right\} = \frac{1}{a} L\{F\left(\frac{t}{a}\right)\}$$

$$L^{-1}\{f(as)\} = \frac{1}{a} F(\frac{t}{a})$$

8.3 EVALUATION OF INVERSE LAPLACE TRANSFORM:

Evaluating the **inverse Laplace transform** involves finding the original time-domain function f(t) from its Laplace transform F(s). Here's how to approach it:

Methods to Evaluate Inverse Laplace Transform:

1) Using Laplace Transform Tables

Most inverse Laplace problems are solved by matching the given F(s) with known Laplace transform pairs.

Common Laplace pairs:

f(t)	$\boldsymbol{F(s) L\{f(t)\}}$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

2) Partial Fraction Decomposition

For rational functions where $F(s) = \frac{P(s)}{Q(s)}$ decompose into simpler fractions.

Example:

$$F(s) = \frac{3s + 1}{(s + 1)(s + 2)}$$

3) Using Shift Theorems

First Shifting Theorem (s-domain shift):

If $L^{-1}\{f(s)\} = F(t)$ then

$$L^{-1}\{f(s - a) = e^{at}F(t)$$

Second Shifting Theorem (time shift):

If f(s) is Laplace Transform of F(t) then

$$L^{-1}\{e^{-as}f(s)\} = G(t)$$

Where $G(t) = \begin{cases} F(t - a), & t > a \\ 0, & t < 0 \end{cases}$

8.4 ELEMENTARY FUNCTION METHOD:

The **Elementary Function Method** for inverse Laplace transforms involves using basic algebraic manipulation and known Laplace transform pairs to find the inverse transform without resorting to complex methods like contour integration or the Bromwich integral.

Here’s a step-by-step guide to this method:

Step 1: Identify Standard Laplace Pairs

Use standard Laplace transform pairs, such as:

$F(s)$	$f(t)$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2 + \omega^2}$	$\frac{\sin(\omega t)}{\omega}$
$\frac{1}{s^2 + \omega^2}$	$\cos(\omega t)$
$\frac{n!}{s^{n+1}}$	t^n

Step 2: Decompose the Function

If the given Laplace transform is a rational function (a ratio of polynomials), decompose it into simpler parts using **partial fraction decomposition**.

Example:

$$F(s) = \frac{3s + 5}{s^2 + 4s + 5}$$

Complete the square in the denominator:

$$s^2 + 4s + 5 = (s + 2)^2 + 1$$

Rewriting the numerator:

$$3s + 5 = 3(s + 2) - 1$$

Split the transform:

$$F(s) = 3.\frac{s + 2}{(s + 2)^2 + 1} - \frac{1}{s + 2)^2 + 1}$$

Using standard pairs, the inverse is:

$$f(t) = 3e^{-2t}\cos(t) - e^{-2t}\sin(t)$$

Step 3: Apply Laplace Shift Theorem

The **first shifting theorem** states:

If $L^{-1}\{f(s)\} = F(t)$ then

$L^{-1}\{f(s-a) = e^{at}F(t)$

Step 4: Use Known Inverse Transforms Directly

Sometimes the transform matches a standard form. For example:

Example:

For $F(s) = \frac{s}{s^2+0}$

Using the pair $\frac{s}{s^2+\omega^2}$<-> Cos(ωt)

f(t)=cos(3t)

Step 5: Use Convolution (if necessary)

If the function is a product of two simpler functions, apply the **convolution theorem**:

$$L^{-1}\{F(s)G(s)\} = (f*g)(t) = \int_0^t f(\tau)\mathrm{g}(\mathrm{t}-\tau)\mathrm{d}\tau$$

8.5 PARTIAL FRACTION METHOD:

The **Partial Fraction Method** is a powerful technique for finding the inverse Laplace transform of rational functions (ratios of polynomials). The method involves decomposing a complex fraction into simpler fractions whose inverse transforms are known.

Steps for the Partial Fraction Method

Given a Laplace transform:

$$F(s) = \frac{P(s)}{Q(s)}$$

where P(s) and Q(s) are polynomials, the steps are:

Step 1: Ensure Proper Fraction

Ensure the degree of P(s) is **less than** the degree of Q(s). If not, perform **polynomial long division** first.

Step 2: Factor the Denominator

Factor Q(s) into its irreducible factors. These can be:

1. **Distinct Linear Factors**: (s−a)
2. **Repeated Linear Factors**: $(s-a)^n$
3. **Irreducible Quadratic Factors**: $s^2 + bs + c$
4. **Repeated Quadratics**: $(s^2 + bs + c)^n$

Step 3: Decompose into Partial Fractions

For each factor type:

1) Distinct Linear Factors:

$$\frac{P(s)}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$$

2) Repeated Linear Factors:

$$\frac{P(s)}{(s-a)^2} = \frac{A}{s-a} + \frac{B}{(s-a)^2}$$

3) Quadratic Factors:

$$\frac{P(s)}{s^2 + bs + c} = \frac{Cs + D}{s^2 + bs + c}$$

Step 4: Solve for Coefficients

Clear the fractions by multiplying both sides by Q(s) and equate coefficients or substitute convenient values of s.

Step 5: Apply Inverse Laplace Transform

Use standard Laplace pairs for each term.

8.6 SOLUTION OF ORDINARY DIFFERENTIAL EQUATION BY USING LAPLACE TRANSFORMATION METHOD:

Solving **ordinary differential equations (ODEs)** using the **Laplace transform method** is a systematic approach that converts the ODE from the time domain into the **s-domain** (complex frequency domain), solves the algebraic equation, and then applies the **inverse Laplace transform** to find the solution in the time domain.

Here's a step-by-step guide for solving ODEs using Laplace transforms:

Step 1: Apply the Laplace Transform

Recall that the Laplace transform of a function f(t) is:

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

For derivatives, the transform rules are:

- **First derivative**:

$$L\{f'(t)\} = F(s) - f(0)$$

- **Second derivative**:

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

And similarly for higher derivatives.

Step 2: Transform the ODE

Convert each term of the ODE using the Laplace transform, incorporating the initial conditions.

Step 3: Solve the Algebraic Equation

Rearrange to solve for F(s), the Laplace transform of the solution.

Step 4: Apply Inverse Laplace Transform

Use known Laplace pairs or partial fractions to find f(t).

Example Problem

Solve the ODE:

y″+3y′+2y=4,y(0)=1,y′(0)=0

Step 1: Apply Laplace Transform

Apply L to both sides:

$$L\{y''\} + 3L\{y'\} + 2L\{y\} = L\{4\}$$

Using the derivative rules:

$$(s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0) + 2Y(s) = \frac{4}{s}$$

Plugging in y(0)=1 and y′(0)=0:

$$(s^2Y(s) - s - (0) + 3(sY(s) - 1) + 2Y(s) = \frac{4}{s}$$

Simplify:

$$(s^2 + 3s + 2)Y(s) - (s + 3) = \frac{4}{s}$$

Rearrange:

$$Y(s) = \frac{s + 3}{(s + 1)(s + 2)} + \frac{4}{s(s + 1)(s + 2)}$$

Step 2: Partial Fraction Decomposition

For $\frac{s+3}{(s+1)(s+2)}$:

Let:

$$\frac{s + 3}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}$$

Multiply both sides by (s+1)(s+2)

$s + 3 = A(s + 2) + B(S + 1)$

Solving for coefficients:

Let $s = -1$

$-1 + 3 = A(1) + B(0) \to 2 = A$

Let s=-2

$-2 + 3 = A(0) + B(-1) \to 1 = -B \to B = -1$

So:

$$\frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{1}{s+2}$$

For $\frac{4}{s(s+1)(s+2)}$ let

$$\frac{4}{s(s+1)(s+2)} = \frac{C}{s} + \frac{D}{s+1} + \frac{E}{s+2}$$

Solving Coefficients:

- Let s=0:

  4=C(1)(2)→ 4=2C → C→C=2

- Let s=−1:

  4=D(−1)(1) → 4=−D → D=−4

- Let s=−2:

  4=E(−2)(−1) → 4=4=2E → E=2

Thus:

$$\frac{4}{s(s+1)(s+2)} = \frac{2}{s} + \frac{4}{s+1} + \frac{2}{s+2}$$

Step 3: Combine and Inverse Transform

$$Y(s) = \frac{2}{s+1} + \frac{1}{s+2} + \frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}$$

Simplify:

$$Y(s) = \frac{2}{s+1} + \frac{1}{s+2} + \frac{2}{s}$$

Inverse Laplace Transform:

- $L^{-1}\frac{2}{s}$=2
- $L^{-1}\frac{-2}{s+1} = -2e^{-t}$
- $L^{-1}\frac{1}{s+2} = -e^{-2t}$

Final solution:

$$y(t) = 2 - 2e^{-t} + e^{-2t}$$

Laplace transforms offer a straightforward method for solving linear ODEs, especially with given initial conditions. If you have more questions or need assistance with another example, feel free to ask!

8.7 SUMMARY:

The **inverse Laplace transform** is a mathematical operation that reverses the Laplace transform, converting a function from the *s-domain* (complex frequency domain) back to the *time domain*. If F(s) is the Laplace transform of a time-domain function f(t), the inverse Laplace transform is denoted as:

$$F(t) = L^{-1}\{f(s)\}$$

8.8 KEY TERMS:

Inverse Laplace Transform-properties, Evaluation of Inverse Laplace Transforms, elementary function method, Partial fraction method, Solution of ordinary differential equation by using Laplace transform method.

8.9 SELF ASSESSMENT QUESTIONS:

1) Explain about the properties of Inverse Laplace transforms
2) Explain about the evaluation of Inverse Laplace transforms
3) Briefly explain about the solution of ordinary differential equation by using Laplace transformation method.

8.10 SUGGESTED READINGS:

1) **"Laplace Transforms and Their Applications" by Allan Pinkus and Samy Zafrany**
   - Focuses on both theoretical aspects and applications.
   - Detailed examples and problems to solidify your understanding.
2) **"Transforms and Applications Handbook" by Alexander D. Poularikas**
   - A detailed guide on various transforms, including Laplace and inverse Laplace transforms.

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LESSON-9

FOURIER SERIES AND FOURIER TRANSFORMS

9.0 AIM AND OBJECTIVE:

The primary aim of this lesson is to understand the concepts of Fourier series,Evaluation of Fourier coefficients, Fourier Transforms, Infinite Fourier Transforms, and Finite Fourier Transforms. After completing this chapter, students should be able to employ various techniques for evaluating residues, including formulas and limit methods; and enhance their analytical and problem-solving skills through the application of the Calculus of Residues.

STRUCTURE:

9.1 Introduction

9.2 Fourier Series

9.3 Evaluation of Fourier Coefficients

9.4 Fourier Transforms-Infinite Fourier Transforms

9.5 Finite Fourier Transforms

9.6 Properties

9.7 Problems

9.8 Summary

9.9 Key Terms

9.10 Self Assessments Questions

9.11 Suggested Books

9.1 INTRODUCTION:

Fourier series represent periodic functions as sums of sines and cosines, with coefficients determined by integrals.Fourier transforms extend this to non-periodic functions, decomposing them into continuous frequency spectra.Infinite Fourier transforms handle functions over infinite domains, while finite transforms apply to functions over finite intervals. Properties like linearity, time-shifting, and frequency-shifting simplify analysis.

9.2 FOURIER SERIES:

Fourier Series is an Infinite Series of a periodic function in terms of Sine and Cosine functions.

If f(x) is a periodic function, then we can express it as an infinite sum of sine and cosine functions as follows:

$$f(x) = \frac{a_0}{2} + a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + \cdots + b_1\sin(x) + b_2\sin(2x)$$

$$+ b_3\sin(3x) + \cdots + b_n\sin(nx) + \cdots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$

Here a_0, a_n and b_n are known as Fourier coefficients. The values of these coefficients are what define the Fourier Series of a function. Constant a_0 is the average value of the periodic function while a_n and b_n are the amplitudes of various sinusoidal functions.

We can calculate a_0, a_n and b_n using the following expressions. For example, if f(x) is a periodic function, then Fourier Coefficients of its Fourier Series in the interval T ≤ x ≤ T+2π are as follows:

$$a_0 = \frac{1}{\pi}\int_T^{T+2T} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_T^{T+2T} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_T^{T+2T} f(x)\sin(nx)dx$$

he equations of a_0, a_n and b_n are known as Euler's Formulae.

In the previous Fourier Series equation, we used both sine and cosine functions. But we can further modify the equation to give an equation only in terms of sinusoids.

We have the term $a_n\cos(nx) + b_n\sin(nx)$ in the equation. We can re-write this as follows:

$$a_n\cos(nx) + b_n\sin(nx) = a_n\sin(nx + 90^{o}) + b_n\sin(nx) = c_n\sin(nx + \theta_n)$$

Where$c_n = \sqrt{a_n^2 + b_n^2}$

and $\theta = tan^{-1}\left(\frac{a_n}{b_n}\right)$

Using these terms, we can derive the sinusoid only Fourier Series Expression of a function as:

$$f(x) = \frac{a_0}{2} + c_1 \sin(x + \theta_1) + c_2 \sin(2x + \theta_2) + c_3 \sin(3x + \theta_3) + \cdots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (c_n \sin(nx + \theta_n))$$

In the above equation, notice that for n = 1, the sinusoidal quantity has the same frequency as the main function (which is 'x' in this case) and it is the Fundamental Frequency of the main waveform. All the subsequent frequencies (for n = 2, n = 3 and so on) are integral multiples of this fundamental frequency which we call as Harmonic Frequencies.

So, for n = 2, the frequency of the corresponding sinusoid is known as Second Harmonic. Similarly, for n = 3, it is Third Harmonic etc.

9.3 EVALUATION OF FOURIER COEFFICIENTS:

From the above discussion, it is clear that the Fourier Coefficients a_0, a_n and b_n are the critical values that we need to calculate for any Fourier Series. We have already seen the expressions for these constants but let us try to derive them.

For this, let us assume that f(x) is a periodic function and its Fourier Series for the interval [T, T+2π] i.e., T ≤ x ≤ T+2π is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx)) + \sum_{n=1}^{\infty} (b_n \sin(nx))$$

Expression for a_0:

In the above equation, let us integrate both sides from x=T to x=T+2π. We get:

$$\int_T^{T+2\pi} f(x)dx = \frac{a_0}{2} \int_T^{T+2\pi} dx + \sum_{n=1}^{\infty} \left[a_n \int_T^{T+2\pi} \cos(nx)dx + b_n \int_T^{T+2\pi} \sin(nx)dx \right]$$

$$= \frac{a_0}{2}(T + 2\pi - T) + \sum_{n=1}^{\infty} \left[a_n \left(\frac{\sin(nx)}{n}\right)_T^{T+2\pi} - b_n \left(\frac{-\cos(nx)}{n}\right)_T^{T+2\pi} \right]$$

$$= a_0\pi + \sum_{n=1}^{\infty} [a_n.0 - b_n.0]$$

$$= a_0\pi$$

From the above equation, we can get the expression for a_0 as:

$$a_0 = \frac{1}{\pi}\int_T^{T+2T} f(x)dx$$

Expression for a_n

Now, consider the original Fourier Series expression once again. Multiply both sides by 'cos(mx)' and integrate the resulting equation from x=T to x=T+2π.

$$\int_T^{T+2\pi} f(x)\cos(mx)\,dx$$

$$= \frac{a_0}{2}\int_T^{T+2\pi}\cos(mx)dx + \sum_{n=1}^{\infty} a_n\left[\int_T^{T+2\pi}\cos(nx)\cos(mx)dx\right]$$

$$+ \sum_{n=1}^{\infty} b_n\left[\int_T^{T+2\pi}\sin(nx)\cos(mx)dx\right]$$

In the above expression, if you observe closely, the integrals corresponding to a_0 and b_n (first and third) are always zero. Coming to the second integral corresponding to a_n, for all m ≠ n cases, it becomes zero and the only possible outcome is for value m = n. Therefore,

$$\int_T^{T+2\pi} f(x)\cos(nx)dx = \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n\int_T^{T+2\pi} 2.\,cos^2(nx)dx\right]$$

$$= \frac{a_n}{2}\int_T^{T+2\pi}\big(1 + \cos(2nx)\big)dx$$

$$= \frac{a_n}{2}\left(x + \frac{\sin(2nx)}{2n}\right)_T^{T+2\pi}$$

$$= \frac{a_n}{2}(2\pi)$$

$$= a_n\pi$$

From the above equation, we can get the expression for a_n as:

$$a_n = \frac{1}{\pi}\int_T^{T+2T} f(x)\cos(nx)dx$$

Expression for b_n

Now, consider the original Series expression once again. Multiply both sides by 'sin(mx)' and integrate the resulting equation from x=T to x=T+2π.

$$\int_T^{T+2\pi} f(x)\sin(mx)\,dx$$

$$= \frac{a_0}{2}\int_T^{T+2\pi} \sin(mx)dx + \sum_{n=1}^{\infty} a_n\left[\int_T^{T+2\pi} \cos(nx)\sin(mx)dx\right]$$

$$+ \sum_{n=1}^{\infty} b_n\left[\int_T^{T+2\pi} \sin(nx)\,sin(mx)dx\right]$$

In the above expression, the integrals corresponding to a_0 and a_n (first and second) are always zero. Coming to the third integral corresponding to b_n, for all m ≠ n cases, it becomes zero and the only possible outcome is for value m = n. Therefore,

$$\int_T^{T+2\pi} f(x)\sin(nx)dx = \frac{1}{2}\sum_{n=1}^{\infty}\left[b_n\int_T^{T+2\pi} 2.\,sin^2(nx)dx\right]$$

$$= \frac{b_n}{2}\int_T^{T+2\pi}\big(1-\cos(2nx)\big)dx$$

$$= \frac{b_n}{2}\left(x - \frac{\sin(2nx)}{2n}\right)_T^{T+2\pi}$$

$$= \frac{b_n}{2}(2\pi)$$

$$= b_n\pi$$

From the above equation, we can get the expression for b_n as:

$$b_n = \frac{1}{\pi}\int_T^{T+2T} f(x)\sin(nx)dx$$

9.4 FOURIER TRANSFORMS - INFINITE FOURIER TRANSFORMS

- **Infinite Fourier sine Transform**

The infinite Fourier sine transform of a function F(x) of x such that $0 < x < \infty$ is denoted by $f_s(n)$, n being a positive integer and is defined as

$$f_s(n) = \int_0^{\infty} F(x) \sin nx \, dx \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Here F(x) is called as the Inverse Fourier sine transform of $f_s(n)$ and defined as

$$F(x) = \frac{2}{\pi}\int_0^{\infty} f_s(n)\sin(nx)dx$$

Thus if $f_s(n) = f_s[F(x)]$, then $F(x) = f_s^{-1}[f_s(n)]$

where f is the symbol for Fourier transform and $f^{-1}$for its inverse.

Infinite Fourier Cosine Transform:

The infinite Fourier Cosine transform of a function F(x) of x such that 0 < x <∞ is defined as

$$f_c(n) = \int_0^{\infty} F(x) \cos nx \, dx, \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

n being a positive integer.

Here the function F(x) is called as the Inverse cosine transform of $f_c(n)$ and is defined as

$$F(x) = \frac{2}{\pi}\int_0^{\infty} f_c(n)\cos(nx)dx$$

Thus if $f_c(n) = f_c[F(x)]$, then $F(x) = f_c^{-1}[f_c(n)]$

where f is the symbol for Fourier transform and f^{-1} for its inverse.

Problems:

1) Find the sine transform of e^{-x}.

We have

$$f_s(n) = \int_0^{\infty} e^{-x} \sin(nx)dx = \left[\frac{e^{-x}}{1+n^2}(-\sin nx - x\cos nx)\right]_0^{\infty} = \frac{\mathrm{n}}{1+n^2}$$

2) Find the cosine transform of x^n e^{-ax}.

We have $\int_0^{\infty} e^{-ax}\cos(nx)dx = \frac{\mathrm{a}}{a^2+n^2}$ and $f_c(n) = \int_0^{\infty} x^n e^{-x}\cos(nx)dx$

Differentiating the first relation n times w.r.t. ‘a’ we find

$$\int_0^{\infty} x^n e^{-x} \cos(nx)dx = (-1)^2 \frac{d^n}{da^n}\left(\frac{\mathrm{a}}{a^2+n^2}\right)$$

$$= \frac{\lfloor n \cos\left\{(n+1)\tan^{-1}\frac{n}{a}\right\}}{(a^2+n^2)^{(n+1)/2}}$$

by usual method.

Hence

$$f_c(n) = \frac{\lfloor n \cos\left\{(n+1)\tan^{-1}\frac{n}{a}\right\}}{(a^2+n^2)^{(n+1)/2}}$$

9.5 FINITE FOURIER TRANSFORMS:

Finite Fourier Sine Transform:

Let f(x) denote a function which is sectionally continuous over the range (0, l). Then the **Finite Fourier Sine transform** of f(x) on this interval is defined as

$$F_s(p) = \overline{f_s}(\text{p}) = \int_0^l f(x)\sin\frac{p\pi x}{l}\,dx$$

where p is an integer (Instead of s, we take p as a parameter)

Inversion Formula for Sine Transform:

If $f_s(p) = F_s$(p) is the finite Fourier sine transform of f(x) in (0, l) then the inversion formula for sine transform is

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty}\overline{f_s}(\text{P})\sin\frac{p\pi x}{l}$$

Proof: For the given function f(x) in (0, l), of we find the half range Fourier sine series, we get,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$\therefore\ b_p = \frac{2}{l}\int_0^l f(x)\sin\frac{p\pi x}{l}\,dx = \frac{2}{l}\overline{f_s}(\text{p})$ by definition

Substituting in (1), we get

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty} f_s(\text{p})\sin\frac{p\pi x}{l}$$

Finite Fourier Cosine Transform:

Let f(x) denote a sectionally continuous function in (0, l). Then the **Finite Fourier cosine transform** of f(x) over (0, l) is defined as

$F_c(p) = \overline{f_c}(p) = \int_0^l f(x) \cos\frac{p\pi x}{l} dx$ where p is an integer.

Inversion Formula for Cosine Transform:

If $\overline{f_c}(P)$ is the finite Fourier cosine transform of f(x) in (0, l), then the inversion formula for cosine transform is

$$f(x) = \frac{1}{l}\overline{f_c}(0) + \frac{2}{l}\sum_{p=1}^{\infty} \overline{f_c}(p) \cos\frac{p\pi x}{l}$$

where $\overline{f_c}(0) = \int_0^l f(x) dx$.

Proof: If we find half range Fourier cosine series for f(x) in (0, l), we obtain,

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ …………… (2)

where $a_n = \frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l} dx$

$\therefore a_p = \frac{2}{l}\overline{f_c}(p)$

$a_0 = \frac{2}{l}\int_0^l f(x) dx = \frac{2}{l}\overline{f_c}(0)$.

Substituting in (2), we get

$$f(x) = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{p=1}^{\infty} f_c(p) \cos\frac{p\pi x}{l}$$

9.6 PROPERTIES:

9.7 PROBLEMS:

1) Find f(x) if itsfinite Fourier sine transform is $\frac{2\pi}{p^3}(-1)^{p-1}$ for p = 1, 2, ……, 0 < x < π.

Sol. By inversion Theorem, we have

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty}\frac{2\pi}{p^3}(-1)^{p-1}\sin px$$

$$= 4\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p^3}\sin px$$

9.8 SUMMARY:

This lesson introduces Fourier analysis, a method for decomposing functions into simpler frequency components. It covers the Fourier series, which represent periodic functions as sums of sines and cosines, and Fourier transforms, extending this to non-periodic functions. We explore calculating Fourier coefficients and properties of infinite and finite Fourier transforms, including linearity and shifting theorems. These tools enable the analysis of signals and solutions to differential equations by transforming functions into the frequency domain, simplifying complex problems through decomposition and manipulation of frequency components.

9.9 TECHNICAL TERMS:

Fourier series - Evaluation of Fourier coefficients - Fourier Transforms - Infinite Fourier Transforms - Finite Fourier Transforms

9.10 SELF-ASSESSMENT QUESTIONS:

1) Find the Fourier sine transform of F(x) = x such that 0 < x < 2. (Question from infinite Fourier transforms.)
2) Find the finite Fourier sine and cosine transforms of

   (i) f(x) = 1 in (0, π)

   (ii) f(x) = x in (0, l)

   (iii) f(x) = 1 in 0$< x < \frac{\pi}{2}$

   = -1 in $\frac{\pi}{2} < x < \pi$

9.11 SUGGESTED BOOKS:

1) M.R. Spiegel ‘Complex VARIABLES’, McGraw-Hill Book Co., 1964.
2) E. Kreyszig ‘Advanced Engineering Mathematics’, Wiley Eastern Pvt., Ltd.1971.
3) H. K. Das & Dr. Rama Varma ‘Mathematical Physics’, S. Chand, 2010.
4) B.D. Gupta ‘Mathematical Physics’, Vikas Publishing House, Sahibabad, 1980.

Prof. G. Naga Raju

LESSON-10

COMPLEX VARIABLES

10.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Complex Variables. The chapter began with understanding of Function of complex number, definition properties, analytic function, Cauchy-Riemann conditions, Polar form, problems. After completing this chapter, the student will understand the complete idea about Complex Variables.

STRUCTURE:

10.1 INTRODUCTION:

Many scientific problems may be treated and solved by methods of complex analysis. These problems can be subdivided into two large classes. The first class consists of elementary problems dealing with electric circuits, vibrating systems, etc., for which the knowledge of complex numbers gained in college Algebra and calculus is sufficient. The second class of problems such as the theory of heat, fluid dynamics, etc., requires a detailed knowledge of the theory of complex analytic functions.

It will be seen that the real and imaginary parts of an analytic function are solutions of Laplace's equation in two independent variables. Consequently, two-dimensional problems can be treated by methods developed in connection with analytic functions. There is, however, a large area of applications in scientific problems in which familiarity with the theory of complex functions beyond this minimum is indispensable.

10.2 SOME BASIC CONCEPTS:

We consider a complex number as having the form a + ib where a and b are real numbers and i, which is called the imaginary number, has the property that $i^2 = -1$. If z = a + ib, then a is called the real part of z and b is called the imaginary part of z and are denoted by Re (z) and

Im (z) respectively. The symbol z stands for a complex variable. The complex conjugate or simply conjugate of z is often denoted by $\bar{z}$ or z^* is given by a – ib. The absolute value or modulus of a complex number or briefly mod z or |z| is given by $|z| = |a + ib| = \sqrt{a^2 + b^2} = |\bar{z}|$. Further $z\bar{z} = \left(\sqrt{a^2+b^2}\right)^2 = |z|^2$ which is an important property.

Since a complex number x + iy can be considered as an ordered pair of real numbers (x, y), we can represent complex numbers by means of the representative points (x, y) in two-dimensionalxy-plane called Argand plane in which x-axis is taken as real axis and y-axis as imaginary axis as shown in figure 1.

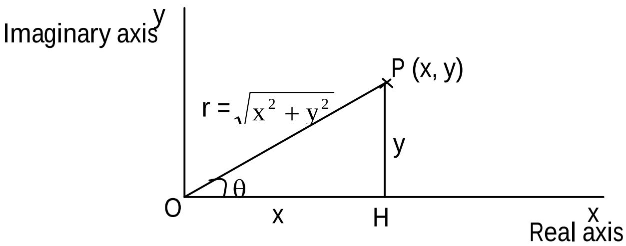

Fig. 10.1: Argand

Further if (r, θ) are the polar coordinates, then x = r cosθ and y = r sinθ. So, the complex number can also be represented as $z = x + iy = r\cos\theta + i\, r\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$ by Euler’s formula.

Consider $z_1 = x_1 + iy_1 = r_1(\cos\theta + i\sin\theta) = r_1e^{i\theta 1}$

where $r_1 = |z_1| = \sqrt{x_1^2 + y_1^2}$ and $\theta_1 = \text{amp } z_1 = \tan^{-1}\dfrac{y_1}{x_1}$ is called the amplitude of z_1 or argument of z_1 (arg z_1).

Similarly consider $z_2 = x_2 + iy_2 = r_2\, e^{i\theta 2}$, Then

$z_1 z_2 = r_1 r_2 e^{i(\theta 1 + \theta 2)}$ in which

$|z_1 z_2| = r_1 r_2 = |z_1| |z_2|$ and

amp $(z_1 z_2)$ = amp z_1 + amp z_2.

(i.e.) modulus of a product of complex numbers is equal to product of the moduli of the individual complex numbers. And amplitude of the product of complex numbers is the sum of the amplitudes of individual complex numbers.

A number ω is called as **nth root** of a complex number z if we write

$$\omega = z^{1/n} = r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \qquad \text{for } k = 0,1,\ldots\ldots n-1$$

In particular, if $z = 1 = 1.e^{i0}$, then

$$\omega = 1^{1/n} = \cos\left(\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$$

$$= 1,\ e^{\frac{2\pi}{n}i}, e^{\frac{4\pi}{n}i}, \ldots\ldots \text{ are the nth roots of unity.}$$

In $w = f(z)$, if to each value of z, there corresponds only one value to w, then w is called a **single** valued function of z.

Example: If $w = z^2$, then for a single value $z = 4$ there corresponds one value to was $4^2 = 16$.

So, $w = z^2$ is single valued. On the otherhand if $w = z^{\frac{1}{2}}$, then for a single value of $z = 4$, there corresponds two values to was + 2 and -2. Thus, its is a double valued or generally called as **many valued function.**

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Q: Show that the modulus of the sum of two complex numbers does never exceed the sum of their moduli.

Solution: Let z_1 and z_2 be the two complex numbers and their conjugates are $\bar{z}_1$ and $\bar{z}_2$

Now $|z_1 + z_2|^2 = (z_1 + z_2)\left(\overline{z_1 + z_2}\right) = (z_1 + z_2)\left(\bar{z}_1 + \bar{z}_2\right) \qquad (\because z^2 = z\bar{z})$

$$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$= |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \overline{z_1 \bar{z}_2}$$

$$= |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(z_1 \bar{z}_2)$$

$$\leq |z_1|^2 + |z_2|^2 + 2 | z_1 \bar{z}_2 | \qquad (\because \operatorname{Re}(z) \leq |z|)$$

or $\leq |z_1|^2 + |z_2|^2 + 2 | z_1| | z_2| \qquad (\because |\bar{z}_2| = | z_2|)$

or $| z_1 + z_2 |^2 \leq (| z_1| + | z_2|)^2$

(i.e.,) $| z_1 + z_2 | \leq (| z_1| + | z_2|)$ ---------- (1)

Q: The modulus of difference of two complex numbers is greater than or equal to the difference of their moduli.

Solution: Let z_1 and z_2 be the two complex numbers and their conjugates are $\bar{z}_1$ and $\bar{z}_2$. Then

$$| z_1 - z_2 |^2 = (z_1 - z_2) \left(\overline{z_1 - z_2}\right) = (z_1 - z_2) \left(\bar{z}_1 - \bar{z}_2\right)$$

$$= |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1 \bar{z}_2)$$

$\geq |z_1|^2 + |z_2|^2 - 2| z_1 \bar{z}_2 | \qquad (\because \operatorname{Re}(z) \leq |z|$ and $-\operatorname{Re}(z) \geq -|z|$

$\geq |z_1|^2 + |z_2|^2 - 2 | z_1| | z_2| \qquad (\because |\bar{z}_2| = | z_2|)$

or $| z_1 - z_2 | \geq (| z_1| - | z_2|)$ ---------- (2)

Note: The inequalities (1) and (2) are important in future lessons on complex variables.

In coordinate geometry, the equation of a circle with origin as center and radius r is given by $x^2 + y^2 = r^2$. This can be represented in complex variables as $|z|^2 = r^2$ or simply $| z| = r$. Thus

$| z| = 1$ represents the equation of a unit circle with origin as centre. Generalizing this concept,

$|z - \alpha| = r$ is the equation of circle with r units radius and centre at α (complex).

Some noteworthy points in understanding the circles are as follows.

$|z - \alpha| = r$: All the points on the circumference of the circle.

$|z - \alpha| < r$: All the points inside the circle.

$|z - \alpha| \leq r$: All the points within and on the circumference of the circle.

$|z - \alpha| > r$: All the points outside the circle.

10.3 DEFINITIONS:

10.3.1 Neighborhood of Point:

It is the set of all points z such that $|z - z_0| < \in$ where $\in$ is an arbitrarily chosen small positive number. i.e., all points interior to $|z - z_0| = \in$ are called the neighborhood of z_0.

10.3.2 Limit:

Let f(z) be defined and single - valued. Let $f(z) = u(x,y) + iv(x,y)$. We say that the number λ is limit of f(z) as z approaches z_0 and write $\underset{z \to z_0}{\mathrm{Lt}} f(z) = \lambda$ if for any arbitrary small positive number $\in$, we can find some positive number δ such that $|f(z) - \lambda| < \in$ for all values in $|z - z_0| < \delta$.This means that the values of f(z) are as close as desired to λ for all z which are sufficiently close to z_0 as shown in figure 2.

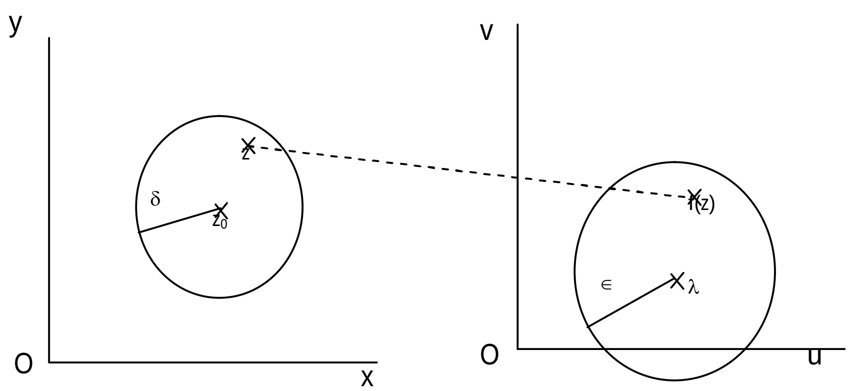

Fig 10.2: **Limit. Dotted line shows the correspondence between z approaching z_0 and f(z) approaching λ**

Note: The definition of a limit implies that in whatever manner z may approach z_0, the limit must be uniquely λ. Since z is a function of x and y (two dimension), z may approach z_0 along any radius vector or any curve. Recalling our concept of a limit in one dimension, $\underset{x \to a}{\mathrm{Lt}} f(x) = k$, it means that the limit from the left and the limit from the right should be equal for the uniqueness of the value k and there are no other paths.

10.3.3 Continuity:

A single valued function f(z) is continuous at the point z_0, if for a given arbitrarily small positive number $\in$, there exists a number δ such that $| f(z) - f(z_0) | < \in$ for all z satisfying $|z - z_0| < \delta$ where δ depends on $\in$.This means that f(z) is continuous at z_0 if $\underset{z \to z_0}{Lt} f(z)$ uniquely exists in whatever manner z approaches z_0 and that value is the value of the function at z_0. Or $\underset{z \to z_0}{Lt} f(z) = f(z_0)$.

10.3.4 Derivatives:

If f(z) is single valued in some region of the z –plane, the derivative of f(z) is defined as

$$f'(z) = \underset{\Delta z \to 0}{Lt} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad \text{----------} \quad (3)$$

provided that the limit exists in whatever manner Δz approaches zero. In such case we say that f(z) is differentiable at z.

10.3.5 Analytic Functions:

A function f(z) which is single valued and differentiable at every point of a region, is said to be analytic in the region. The terms regular and holomorphic are sometimes used as synonyms for analytic.

10.3.6 Singular Points:

A point at which f(z) fails to be analytic is called a singular point or singularity of f(z). We consider various types of singularities that exist at a latter stage.

Note: The practical approach in finding out the singular point is to find out the point where the given function becomes infinite.

10.4 Complex Variables:

Complex variables are variables that can take on complex numbers as values. A complex number is a number that can be expressed in the form z = x + iy, where x and y are real numbers, and i is the imaginary unit, which satisfies $i^2 = -1$.

Definition of Complex Variables:

A complex variable z is typically expressed in the form:

$$z = x + iy$$

where x and y are real numbers, and i is the imaginary unit,

which satisfies $i^2 = -1$.

10.5 FUNCTIONS OF COMPLEX VARIABLES:

There are various types of functions that involve complex variables. Some of the functions with complex variables are Analytical Functions and Elementary Functions. Some of the common elementary functions are: Exponential Functions, Trigonometric Functions, Logarithmic Functions, and Power Functions.

10.5.1 Analytic Functions:

A function f(z) is said to be analytic (or holomorphic) at a point z_0 if it is differentiable at z_0 and in a neighbourhood around z_0. If a function is analytic at every point in its domain, it is called an entire function.

10.5.2 Elementary Functions:

Exponential Function	**Exponential function for a complex variable z.**	$\mathbf{e^z = e^{x+yi} = e^x(\cos y + i \sin y)}$
Trigonometric Functions	Sine and cosine functions for a complex variable z.	$\sin z = (e^{iz} - e^{-iz})/2$ $\cos z = (e^{iz} + e^{-iz})/2$
Logarithmic Function	Logarithm of a complex number z = reiθ (in polar form).	$\log z = \log r + i\theta$
Power Functions	Power of a complex number z raised to an integer n.	$z^n = (re^{i\theta})^n = r^n e^{in}$

10.6 CAUCHY-RIEMANN CONDITIONS:

A necessary condition that w = f(z) = u (x, y) + iv (x, y) be analytic in a region R is that u and v satisfy the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ ---------- } (4)$$

$$\text{or} \qquad ux = vy \text{ and } uy = -vx$$

In addition to the existence of the partial derivatives in (4), if they are also continuous, then the Cauchy – Riemann equations are sufficient conditions for f(z) to be analytic in R.

Solution:

Necessary: If f(z) = u (x, y) + iv (x, y) is to be analytic, the limit

$$\underset{\Delta Z \to 0}{\text{Lt}} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z)$$

$$= \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{\text{Lt}} \frac{[u(x+\Delta x, y+\Delta y)+iv(x+\Delta x, y+\Delta y)]-[u(x, y)+iv(x, y)]}{\Delta x + i\,\Delta y}$$

-------------- (5)

must exist in whatever manner Δz or (Δx and Δy) tends to zero. Let us consider two simple approaches

Case 1: In $\Delta z = \Delta x + i\Delta y$ approaching zero let us consider that $\Delta y = 0$ which means that $\Delta z = \Delta x$ (purely real). So Δz tending to zero means it approaches zero along the real axis. In such a case, (5) becomes

$$f'(z) = \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{u(x+\Delta x, y)-u(x, y)}{\Delta x} + i\,\frac{v(x+\Delta x, y)-v(x, y)}{\Delta x}\right]$$

$$= u_x + iv_x \qquad \text{---------- (6)}$$

provided the partial derivative exist.

Case 2: If $\Delta x = 0$ and $\Delta y \to 0$, then $\Delta z = \Delta y$ (purely imaginary) tends to zero. So (5) becomes

$$f'(z) = \underset{\Delta y \to 0}{\text{Lt}} \left[\frac{u(x, y+\Delta y)-u(x, y)}{i\,\Delta y} + \frac{v(x, y+\Delta y)-v(x, y)}{\Delta y}\right]$$

$$= \frac{1}{i}u_y + v_y = -\,iu_y + v_y \qquad \text{---------- (7)}$$

Now f(z) cannot be analytic unless these two limits as in (6) and (7) must be identical. So the necessary condition that f(z) be analytic is

$u_x + iv_x = -\,iu_y + v_y$

or $\quad u_x = v_y\,; v_x = -\,iu_y$ ---------- (8)

Sufficient:

Apart from the existence of the partial derivatives in (8), since u_x and u_y are supposed continuous, we have

$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$

$= \{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)\} + \{u(x, y + \Delta y) - u(x, y)\}$

$= (u_x + \epsilon_1)\,\Delta x + (u_y + \eta_1)\,\Delta y$ by mean value theorem

$= u_x\Delta x + u_y\Delta y + \in_1\Delta x + \eta_1\Delta y$

where $\in_1$ and η_1 tend to zero as $\Delta x\to 0$ and $\Delta y\to 0$

Again, considering that v_x and v_y are supposed continuous, we get a similar expression for Δv as$\Delta v = v_x\Delta x + v_y\Delta y + \in_2\Delta x + \eta_2\Delta y$ where $\in_2$, η_2 tend to zero as Δx and Δy tend to zero.

Then $\Delta w = \Delta u + i\Delta v$

$$= (u_x + iv_x)\,\Delta x + (u_y + iv_y)\,\Delta y + \in\Delta x + \eta\Delta y \qquad \text{----------} \quad (9)$$

where $\in = \in_1 + i\in_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x\to 0$ and $\Delta y\to 0$.

If $\Delta w = f(z)$ satisfies Cauchy – Riemann equations then we have to prove that unique derivative of f(z) exists.

By Cauchy – Riemann equations, (9) takes the form

$\Delta w = (u_x + iv_x)\,\Delta x + (-v_x + iu_x)\,\Delta y + \in\Delta x + \eta\Delta y$

$$= (u_x + iv_x)\,(\Delta x + i\Delta y) + \in\Delta x + \eta\Delta y$$

Dividing with $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \underset{\Delta Z\to 0}{\text{Lt}} \frac{\Delta w}{\Delta z} = u_x + iv_x \text{----------} \quad (10)$$

so that the derivative exists and unique. That is f(z) is analytic.

10.6.1 Examples

(1) Show that $f(z) = \bar{z}$ is nowhere analytic

Solution: If $f(z) = \bar{z} = x - iy$, then

$f(z + \Delta z) = \bar{z} + \overline{\Delta z} = (x - iy) + (\Delta x - i\Delta y)$

$$\therefore \frac{d\bar{z}}{dz} = \underset{\Delta Z\to 0}{\text{Lt}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \underset{\substack{\Delta x\to 0\\ \Delta y\to 0}}{\text{Lt}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Let Δx and Δy approach along the radius vector y = mx. Then

$$\frac{d\bar{z}}{dz} = \underset{\Delta x \to 0}{Lt} \frac{\Delta x - i\,m\,\Delta x}{\Delta x + i\,m\,\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{1 - i\,m}{1 + i\,m}$$

$$= \frac{1 - i\,m}{1 + i\,m}.$$

This value is not unique since m is an arbitrary constant. So $\frac{d\bar{z}}{dz}$ does not exist. Hence it is nowhere analytic.

(2) Prove that the function u + iv = f(z) where

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} & (z \neq 0) \\ 0 & (z = 0) \end{cases}$$

Is continuous and that the Cauchy – Riemann equation are satisfied at the origin'

Yet f ′(0) = does not exist.

Solution: $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$

$$= \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2} \quad \text{from which}$$

$$u = \frac{x^3 - y^3}{x^2 + y^2} \text{ and } v = \frac{x^3 + y^3}{x^2 + y^2} \quad \text{when } z \neq 0.$$

Both u and v are rational and finite for all values of z ≠ 0. Hence f(z) is continuous for all z ≠ 0. Now at z = 0, both u and v are zero. So, they are continuous at the origin.

We know that

$$\frac{\partial u}{\partial x} = \underset{h \to 0}{Lt} \frac{u(x+h, y) - u(x, y)}{h}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{h \to 0}{Lt} \frac{u(h, 0) - u(0, 0)}{h}$$

$$= \underset{h \to 0}{\mathrm{Lt}} \frac{h - 0}{h} = 1 \qquad (\because \text{at } z = 0,\ f(0) = 0)$$

Similarly, it is seen that

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=0\\y=0}} = \underset{h \to 0}{\mathrm{Lt}} \frac{-h - 0}{h} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{h \to 0}{\mathrm{Lt}} \frac{h - 0}{h} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{\substack{x=0\\y=0}} = \underset{h \to 0}{\mathrm{Lt}} \frac{h - 0}{h} = 1$$

Thus, at the origin $u_x = v_y$ and $u_y = -v_x$ (i.e) Cauchy-Riemann equation are satisfied. But the derivative at the origin is

$$f'(0) = \underset{Z \to 0}{\mathrm{Lt}} \frac{f(z) - f(0)}{z - 0}$$

$$= \underset{\substack{x \to 0\\y \to 0}}{\mathrm{Lt}} \frac{x^3 - y^3}{(x^2 + y^2)(x + i\,y)} + i \frac{x^3 + y^3}{(x^2 + y^2)(x + i\,y)}$$

Since both numerator and denominator are homogeneous expressions of the same order, let x and y approach zero along any radius vector (i.e.) $y = mx$. Then

$$f'(0) = \underset{x \to 0}{\mathrm{Lt}} \frac{1 - m^3}{(1 + m^2)(1 + im)} + i \frac{1 + m^3}{(1 + m^2)(1 + im)}$$

which is independent of x. Further, since m is arbitrary, $f'(0)$ is not unique and f(z) is continuous everywhere.

(3) Show that $f(z) = \sqrt{|xy|}$ is not analytic at the origin although Cauchy-Riemann equations are satisfied at that point.

Solution: Given that $f(z) = \sqrt{|xy|}$. Since |xy| is always a positive quantity, $f(z) = \sqrt{|xy|}$ is always real. So $u(x, y) = \sqrt{|xy|}$; $v(x, y) = 0$.

Now $\left(\frac{\partial u}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{h \to 0}{\text{Lt}} \frac{u(h,0) - u(0,0)}{h} = \underset{h \to 0}{\text{Lt}} \frac{0 - 0}{h} = 0$

Similarly, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are zeros. So, Cauchy – Riemann equation are satisfied at the origin.

Now $f'(0) = \underset{Z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z - 0} = \underset{\substack{x \to 0\\y \to 0}}{\text{Lt}} \frac{\sqrt{|xy|}}{x + iy}$

Since the numerator and denominator are homogeneous expressions of the same order, consider the radius vector y = mx along which x and y approach zero. Then $f'(0) = \underset{x \to 0}{\text{Lt}}$

$\frac{x\sqrt{|m|}}{x(1+im)} = \frac{\sqrt{|m|}}{(1+im)}$ which gives different values for different values of the arbitrary constant m. Hence $f'(0)$ is not unique or the derivative does not exist or f(z) is not analytic at the origin.

(4) Show that $w = x^2 - y^2 + 2\,ixy$ is everywhere analytic in the entire complex plane and express the derivative of w w.r.t z as a function of z alone.

Solution: Given that $w = (x^2 - y^2) + 2\,ixy$in which $u = x^2 - y^2$ and $v = 2\,xy$

$\therefore u_x = 2x,\ u_y = -2y;\quad v_x = 2y,\ v_y = 2x$

(i.e.) Cauchy-Riemann equations are identically satisfied in the complex plane. More over the first order partial derivatives are everywhere continuous. So the derivative $\frac{dw}{dz}$ should exist according to the sufficient condition for the analytic functions and it is given by

$$\frac{dw}{dz} = u_x + iv_x \qquad \text{----------} \quad (10)$$

$$= 2x + 2iy = 2(x + iy) = 2z.$$

(5) In any analytic function w = u(x, y) + iv(x, y), if x and y are replaced by their equivalents, $x = \frac{z + \bar{z}}{2}$ and $y = \frac{z - \bar{z}}{2i}$ then w will appear as a function of z alone.

Solution: Although z and $\bar{z}$ are clearly dependent, w can be formally considered as a function two new independent variables z and $\bar{z}$. Then, if w has to appear as a function of z only, we have to prove that $\frac{\partial w}{\partial \bar{z}}$ is identically zero.

Now

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial (u+iv)}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

But from the expression of x and y in terms of z and $\bar{z}$,

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \quad ; \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i} = \frac{i}{2}$$

So

$$\frac{\partial w}{\partial \bar{z}} = \left(\frac{1}{2}u_x + \frac{i}{2}u_y\right) + i\left(\frac{1}{2}v_x + \frac{i}{2}v_y\right)$$

$$= \frac{1}{2}\left(u_x - v_y\right) + \frac{i}{2}\left(u_y + v_x\right) = 0$$

($\because$ w = u + iv is given as analytic and so u and v satisfy Cauchy-Riemann equation) (i.e.) w is a function of z alone.

10.7 POLAR FORM OF CAUCHY – RIEMANN EQUATIONS:

The polar form of the Cauchy – Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial r} = \frac{-1}{r}\frac{\partial v}{\partial \theta}$$

Proof: In the case of polar form of complex numbers x = r cosθ and y = r sinθ

$$x^2 + y^2 = r^2;\ \tan\frac{y}{x} \Rightarrow \theta = tan^{-1}\frac{y}{x}$$

$$2x\partial x = 2r\partial r \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= \frac{r\cos\theta}{r} = \cos\theta$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$

And from $\theta = tan^{-1}\frac{y}{x}$,

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

$$= \frac{-x^2 y}{x^2+y^2} \times \frac{1}{x^2}$$

$$= \frac{-y}{x^2+y^2}$$

$$= \frac{-r\sin\theta}{r^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{-\sin\theta}{r}$$

Similarly $\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x^2}\right)$

$$= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

Hence

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}cos\theta - \frac{\partial u}{\partial\theta}\frac{sin\theta}{r}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$

$$= \frac{\partial u}{\partial r} sin\theta + \frac{\partial u}{\partial \theta}\frac{cos\theta}{r}$$

Similarly

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} cos\theta - \frac{\partial v}{\partial \theta}\frac{sin\theta}{r}$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} sin\theta + \frac{\partial v}{\partial \theta}\frac{cos\theta}{r}$$

From Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$$\frac{\partial u}{\partial r} cos\theta - \frac{\partial u}{\partial \theta}\frac{sin\theta}{r} = \frac{\partial v}{\partial r} sin\theta + \frac{\partial v}{\partial \theta}\frac{cos\theta}{r} \qquad \text{------------ (1)}$$

$$\frac{\partial u}{\partial r} sin\theta + \frac{\partial u}{\partial \theta}\frac{cos\theta}{r} = -\frac{\partial v}{\partial r} cos\theta + \frac{\partial v}{\partial \theta}\frac{sin\theta}{r} \qquad \text{------------ (2)}$$

Multiplying eq. (1) by cosθ and eq. (2) by sinθ and then adding them gives

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{------------ (3)}$$

Multiplying eq. (1) by -sinθ and eq. (2) by cosθ and then adding them gives

$$\frac{\partial u}{\partial r} = \frac{-1}{r}\frac{\partial v}{\partial \theta} \qquad \text{------------ (4)}$$

Equations (3) and (4) are called Polar form of Cauchy-Riemann Equations.

10.8 SUMMARY:

This lesson, starting with an introduction, projects the rudiments of complex numbers and functions. Then the basic definitions of certain parameters already familiar in real analysis are given with respect to complex region. Uniqueness of the limit is highlighted which can be appreciated while dealing with the derivation of Cauchy-Riemann conditions. The equation of circle and inequalities in the complex plane, play important role in future theorems and problems. The definition of an analytic function is given and the necessary and sufficient conditions for a function to be analytic are derived. The real and imaginary parts of every analytic function are seen to be harmonic functions (conjugates) satisfying Laplace equation.

Typical and assorted problems have been worked and questions given at the end of the lesson.

10.9 KEY TERMS:

Argand diagram - mod.z - amp.z - polar form - single valued function - neighborhood - limit - continuity - differentiability - Analytic functions - Cauchy - Riemann equations.

10.10 SELF-ASSESSMENT QUESTIONS:

1) If f(z) = u + iv is an analytic function where u^2+v^2 is a constant, show that f(z) is a constant.
2) Show that $w = z\bar{z}$ is everywhere continuous and it is nowhere analytic except at the origin.
3) If $z = re^{i\theta}$, show that the Cauchy-Riemann equations take the form $u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$
4) If f(z) and $\overline{f(z)}$ are both analytic functions show that f(z) is a constant.
5) Determine the analytic function f(z) = u + iv when $u + v = x^2 - y^2 + 2xy$.

10.11 REFERENCE BOOKS:

1) M.R. Spiegel ‘Complex variables’, McGraw-Hill Book co., 1964.
2) E. Kreyszig ‘Advanced engineering Mathematics’, Wiley Eastern Pvt., Ltd., 1971.
3) B.D. Gupta ‘Mathematical Physics’, Vikas publishing House, Sahibabad, 1980.

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LESSON-11

CAUCHY'S INTEGRAL THEOREM

11.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Cauchy's Integral Theorem. The chapter began with understanding of Cauchy's integral theorem, Cauchy's integral formula, problems. After completing this chapter, the student will understand the complete idea about Cauchy's Integral theorem.

STRUCTURE:

11.1 INTRODUCTION:

Cauchy's integral formula is a central statement in complex analysis in mathematics. It expresses that a holomorphic function defined on a disk is determined entirely by its values on the disk boundary. For all derivatives of a holomorphic function, it provides integration formulas. Also, this formula is named after Augustin-Louis Cauchy. In this article, you will learn Cauchy's Integral theorem and the formula with the help of solved examples.

Before going to the theorem and formula of Cauchy's integral, let's understand what a simply connected region is.

Simply connected Region:

A connected region is called a simply connected region, if all the interior points of a closed curve C, are illustrated in region D, are also the points of region D.

11.2 CAUCHY’S INTEGRAL THEOREM:

Statement: If a function f(z) is analytic and its derivative f'(z) is continuous at each point within and on a closed-curve C then

$$\int_c f(z)\, dz = 0$$

Proof: Let the region enclosed by the closed curve C be R. And we know

$$f(z) = u + iv;\ z = x + iy;\ dz = dx + idy$$

Taking L.H.S. from the statement

$$\int_c f(z)\, dz = \int_c (u + iv)\,(dx + idy)$$

$$= \int_c (udx + vdy) + i\int_c (vdx + udy)$$

By Green’s theorem, we know that

$$\int_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\Rightarrow \int_c f(z)\, dz = \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \qquad \text{------------ (1)}$$

By Cauchy – Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \qquad \text{------------ (2)}$$

From eq. (1) and eq. (2), we get

$$\int_c f(z)\, dz = 0 + i(0)$$

$$= 0 \qquad \text{(which is R.H.S. in the statement)}$$

Hence $\int_c f(z)\, dz = 0$

11.3 CAUCHY’S INTEGRAL FORMULA:

Statement: If a complex function f(z) is analytic within and on a closed contour c inside a simply-connected domain, and if z_0 is any point in the middle of C, then

$$f(z_o) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z - z_o)}\, dz$$

Here, the integral should be taken in the positive sense around c.

Proof: Consider the function $\frac{f(z)}{(z-z_o)}$ is analytic at all points in C except z = z_0. Now with the point a z_0 as the centre draw a small circle C_1 of radius r lying entirely within C. Now $\frac{f(z)}{(z-z_o)}$ is analytic in the region between C and C_1, hence by Cauchy's Integral theorem for multiply connected regions, we have

$$\int_C \frac{f(z)}{(z-z_o)}dz = \int_{C_1} \frac{f(z)}{(z-z_o)}dz$$

$$= \int_{C_1} \frac{f(z)-f(z_o)+f(z_o)}{(z-z_o)}dz$$

$$= \int_{C_1} \frac{f(z)-f(z_o)}{(z-z_o)}dz + \int_{C_1} \frac{f(z_o)}{(z-z_o)}dz \quad \text{------------ (3)}$$

For any point on C_1

$$\int_{C_1} \frac{f(z)-f(z_o)}{(z-z_o)}dz$$

From complex circle equation $z - z_o = re^{i\theta}$

$$z = z_o + re^{i\theta}$$

$$dz = ire^{i\theta}d\theta$$

$$\int_{C_1} \frac{f(z)-f(z_o)}{(z-z_o)}dz = \int_{C_1} \frac{f\left(z_o + re^{i\theta}\right)-f(z_o)}{re^{i\theta}}ire^{i\theta}d\theta$$

$$= i\int_{C_1}\left[f\left(z_o + re^{i\theta}\right)-f(z_o)\right]d\theta$$

When r→0

$$\int_{C_1} \frac{f(z)-f(z_o)}{(z-z_o)}dz = 0$$

Now $$\int_{C_1} \frac{dz}{(z-z_o)} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}d\theta$$

$$= \mathrm{i}[\theta]_0^{2\pi}$$

$$\int_{C_1} \frac{dz}{(z - z_o)} = 2\pi i$$

Substitute above values in eq. (3)

$$\int_C \frac{f(z)}{(z - z_o)} dz = 0 + f(z_o)$$

$$\therefore f(z_o) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_o)} dz$$

11.3.1 Generalisation of Cauchy's Integral Formula:

If f(z) is an analytic function within and on a simple closed curve C and if z_0 is any point within c, then

$$f^n(z_o) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z - z_o)^{n+1}} dz$$

11.4 CONVERSE OF CAUCHY'S INTEGRAL THEOREM:

If a complex function f(z) is continuous throughout the simple connected domain D and if $\int_c f(z)\, dz = 0$ for every closed contour c in D, then f(z) will be an analytic function in D.

This theorem is also known as **Morera's theorem**.

Example: Evaluate $\int_c \frac{z^2}{(z-5)} dz$, where "c" is the circle such that $|z| = 2$.

Solution: Comparing $\int_c \frac{z^2}{(z-5)} dz$ with $\int_c f(z) dz$, we get;

$f(z) = z^2/(z - 5)$

This function is not analytic at $z = 5$.

However, this point lies outside the circle defined by $|z| = 2$.

Therefore, f(z) is an analytic function at all points inside and on the closed curve c.

Thus, by Cauchy's theorem, we can write $\int_c f(z)\, dz = 0$.

11.5 SUMMARY:

This lesson introduces Cauchy's Integral Theorem, stating that the integral of an analytic function over a closed contour in a simply connected domain is zero, and Cauchy's Integral

Formula, which allows us to calculate the value of an analytic function and its derivatives at a point inside the contour using a contour integral; these theorems are essential for understanding complex function behavior and evaluating integrals, forming the basis for advanced complex analysis concepts.

11.6 KEY TERMS:

Cauchy's Integral Theorem- Cauchy's Integral.

11.7 SELF-ASSESSMENT QUESTIONS:

11.8 REFERENCE BOOKS:

1) M.R. Spiegel-‘Complex variables’, McGraw – Hill Book co., 1964.

2) E. Kreyszig-‘Advanced engineering mathematics’, Wiley Eastern Pvt., Ltd., 1971.

3) B.D. Gupta-‘Mathematical Physics’, Vikas publishing House, Sahibabad, 1980.

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LESSON-12

TAYLOR'S AND LAURENT'S EXPANSIONS

12.0 AIM AND OBJECTIVE:

The primary goal of this chapter is to understand the concept of Taylor's and Laurent's Expansions. The chapter began with understanding of Taylor's Series-Laurent's expansions and Problems. After completing this chapter, the student will understand the complete idea about Taylor's and Laurent's Expansions.

STRUCTURE:

12.1 Introduction

12.2 Taylor's Series

12.3 Laurent's Expansion

12.4 Problems

12.5 Summary

12.6 Key Terms

12.7 Self-Assessment Questions

12.8 Reference Books

12.1 INTRODUCTION:

Taylor's Theorem provides a way to approximate a function near a specific point using its derivatives at that point. It expresses the function as a sum of terms involving these derivatives, multiplied by powers of the difference between the input and the point of expansion.Laurent's Theorem generalizes Taylor's Theorem to functions that may have singularities (points where they are not well-behaved). It represents such functions as a series involving both positive and negative powers.

Taylor's theorem is extremely useful in numerical analysis, optimization, and physics for approximating functions and solving differential equations and Laurent's theorem is crucial in complex analysis for studying the behavior of functions near singularities, classifying singularities, and evaluating complex integrals.

12.2 TAYLOR'S SERIES:

Statement: If a function f(z) is analytic at all points inside and on the circle C with centre a and radius r then each point z inside C is such that $|z-a| \leq r$ then

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''}{2}(z-a)^2 + \cdots + \frac{fn}{2}(z-a)^n + \cdots$$

Proof: Let C be the circle with centre a and r be the radius and take a point z within C and draw a circle C_1 with the same centre enclosing the point z.

Let w be the any point on C_1. The magnitude of (z-a) is smaller than the magnitude of (w-a).

$$|z-a| < |w-a|$$

$$\frac{|z-a|}{|w-a|} < 1$$

Now $$\frac{1}{w-z} = \frac{1}{w-a+a-z}$$

$$= \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(w-a)\left[1-\frac{z-a}{w-a}\right]}$$

$$= \frac{1}{(w-a)}\left[1-\frac{z-a}{w-a}\right]^{-1}$$

$$(1-x)^{-1} = 1 + x + x^2 + \cdots \text{ when } |x|<1$$

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1 - \frac{z-a}{w-a} + \frac{(z-a)^2}{(w-a)^2} + \cdots + \frac{(z-a)^n}{(w-a)^n} + \cdots\right]$$

$$\frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^n}{(w-a)^{n+1}} + \cdots$$

It is a uniformly convergent series i.e., it is integrable hence multiplying both sides with f(w) and then integrating with respect to w over C_1.

$$\int_{C_1} \frac{f(w)}{w-z}dw$$

$$= \int_{C_1} \frac{f(w)}{w-z}dw + (z-a)\int_{C_1} \frac{f(w)}{(w-z)^2}dw$$

$$+ (z-a)^2\int_{C_1} \frac{f(w)}{(w-z)^3}dw + \cdots + (z-a)^n\int_{C_1} \frac{f(w)}{(w-z)^{n+1}}dw + \cdots$$

----- (1)

We know that from Cauchy integral formula for derivatives

i) $\int_{C_1} \frac{f(w)}{w-z} dw = 2\pi i\, f(a)$

ii) $\int_{C_1} \frac{f(w)}{(w-z)^2} dw = 2\pi i\, f'(a)$

iii) $\int_{C_1} \frac{f(w)}{(w-z)^{n+1}} dw = \frac{2\pi i}{n!} f^n(a)$

Now eq. (1) becomes

$$2\pi i\, f(z) = 2\pi i\, f(a) + (z-a)2\pi i\, f'(a) + \cdots + (z-a)^n \frac{2\pi i}{n!} f^n(a) + \cdots$$

Dividing with 2πi on both sides gives

$$f(z) = f(a) + (z-a)f'(a) + \cdots + (z-a)^n \frac{1}{n!} f^n(a) + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \ \ where\ a_n = \frac{f^n(a)}{n!}$$

12.3 LAURENT'S EXPANSION:

Statement: If f(z) is analytic inside and on the boundary of ring shaped region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 ($r_1 > r_2$) respectively and with the centre a then for all z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} b_n (z-b)^{-n}$$

$$where\ a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} dw\ where\ n = 0,1,2, \ldots$$

$$where\ b_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} dw\ where\ n = 1,2,3, \ldots$$

Proof: Let C_1 and C_2 are two circles with centrea of radii r_1 and r_2 ($r_1 > r_2$) in an anti-clockwise direction. Given that f(z) is analytic on C_1 and C_2. Let γ be the closed path containing ABCDBAEFGA then f(z) is analytic within and on the curve γ. Hence by Cauchy Integral Formula we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-a}dw$$

where z is any point in the region R.

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-a}dw + \frac{1}{2\pi i}\int_{AB}\frac{f(w)}{w-a}dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-a}dw + \frac{1}{2\pi i}\int_{BA}\frac{f(w)}{w-a}dw$$

But $\frac{1}{2\pi i}\int_{AB}\frac{f(w)}{w-a}dw + \frac{1}{2\pi i}\int_{BA}\frac{f(w)}{w-a}dw = 0$

$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-a}dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-a}dw$ ……..

(1)

Since $\frac{f(w)}{w-a}$ is analytic and in this case w lies on C_1

$$|z-a| < |w-a|$$

$$\frac{|z-a|}{|w-a|} < 1$$

Now $\frac{1}{w-z} = \frac{1}{w-a+a-z}$

$$= \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(w-a)\left[1-\frac{z-a}{w-a}\right]}$$

$$= \frac{1}{(w-a)}\left[1-\frac{z-a}{w-a}\right]^{-1}$$

$(1-x)^{-1} = 1 + x + x^2 + \cdots$ when $|x|<1$

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1-\frac{z-a}{w-a}+\frac{(z-a)^2}{(w-a)^2}+\cdots+\frac{(z-a)^n}{(w-a)^n}+\cdots\right]$$

$$\frac{1}{w-z} = \frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\cdots+\frac{(z-a)^n}{(w-a)^{n+1}}+\cdots$$

Multiplying both sides with $\frac{f(w)}{2\pi i}$ and then integrate with respect to w over C_1

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}dw$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-a}dw + \frac{(z-a)}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^2}dw$$

$$+ \frac{(z-a)^2}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^3}dw + \cdots + \frac{(z-a)^n}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw + \cdots$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \qquad \text{----- (2)}$$

Now consider

$$\int_{C_2}\frac{f(w)}{w-a}dw;$$

in this case w lies on circle C_2 then

$$|w-a| < |z-a|$$

$$\frac{|w-a|}{|z-a|} < 1$$

Now

$$\frac{1}{w-z} = \frac{1}{w-a+a-z}$$

$$= \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{-(z-a)\left[1-\frac{w-a}{z-a}\right]}$$

$$= \frac{1}{-(z-a)}\left[1-\frac{w-a}{z-a}\right]^{-1}$$

$$\frac{1}{w-z} = \frac{-1}{z-a}\left[1+\frac{w-a}{z-a}+\frac{(w-a)^2}{(z-a)^2}+\cdots+\frac{(w-a)^n}{(z-a)^n}+\cdots\right]$$

$$\frac{1}{w-z} = -\left[\frac{1}{z-a}+\frac{w-a}{(z-a)^2}+\frac{(w-a)^2}{(z-a)^3}+\cdots+\frac{(w-a)^n}{(z-a)^{n+1}}+\cdots\right]$$

Multiplying both sides with $\frac{-f(w)}{2\pi i}$and then integrate with respect to w over C_2

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw$$

$$=\frac{(z-a)^{-1}1}{2\pi i}\int_{C_2}f(w)dw+\frac{(z-a)^{-2}}{2\pi i}\int_{C_2}f(w)(w-a)dw+\cdots$$

$$+\frac{(z-a)^{-n}}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{-n}}dw+\cdots$$

$$=b_1(z-a)^{-1}+b_2(z-a)^{-2}+\cdots \qquad \text{----- (3)}$$

Substitute equations (2) and (3) in equation (1), we get

$$f(z)=[a_0+a_1(z-a)+a_2(z-a)^2+\cdots]+[b_1(z-a)^{-1}+b_2(z-a)^{-2}+\cdots]$$

$$\therefore f(z)=\sum_{n=0}^{\infty}a_n\,(z-a)^n+\sum_{n=0}^{\infty}b_n\,(z-b)^{-n}$$

12.4 PROBLEMS:

1) Find the first four terms of the Taylor series expansion of the complex variable function $f(z)=\frac{z+1}{(z-3)(z-4)}$ about z = 2 find the origin of convergence.

Solution: By Taylor series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''}{2!}(z-a)^2+\cdots+\frac{f'''}{3!}(z-a)^3+\cdots$$

Since z = 2

$$f(2)=\frac{2+1}{(2-3)(2-4)}=\frac{3}{2}$$

$$f(z)=\frac{-4}{(z-3)}+\frac{5}{(z-4)}$$

$$f'(z)=\frac{4}{(z-3)^2}+\frac{(-5)}{(z-4)^2}$$

$$f'(2) = \frac{4}{1} - \frac{5}{4} = \frac{11}{4}$$

$$f''(z) = \frac{-8}{(z-3)^3} + \frac{10}{(z-4)^3}$$

$$f''(2) = \frac{27}{4}$$

$$f'''(z) = \frac{24}{(z-3)^4} + \frac{(-30)}{(z-4)^4}$$

$$f'''(2) = \frac{177}{8}$$

$$\therefore f(z) = \frac{z+1}{(z-3)(z-4)}$$

$$f(z) = \frac{3}{2} + \frac{11}{4}(z-2) + \frac{27}{4 \times 2}(z-2)^2 + \frac{177}{8 \times 2 \times 3}(z-2)^3$$

$$f(z) = \frac{1}{2}\left[3 + \frac{11}{2}(z-2) + \frac{27}{4}(z-2)^2 + \frac{177}{24}(z-2)^3\right]$$

12.5 SUMMARY:

This lesson explores power series representations of complex functions, beginning with Taylor's Theorem, which expresses an analytic function as a convergent power series in a neighborhood of a point, effectively generalizing the real-variable Taylor expansion. The proof highlights the role of Cauchy's integral formula. Laurent's Theorem then extends this concept to functions with singularities, providing a series representation involving both positive and negative powers of $(z-z_0)$ within an annulus surrounding the singularity. The derivation emphasizes the importance of contour integration and residue calculus.

12.6 KEY TERMS:

Taylor's Theorem-Laurent's Theorem.

12.7 SELF-ASSESSMENT QUESTIONS:

1) Find the first four terms of Taylor series of expansion of a complex variable function $f(z) = \frac{1}{(z-1)(z-3)}$ about the point z = 4 find the origin of convergence.

2) Expand Laurent's series $f(z) = \frac{1}{(z-1)(z-2)}\ for\ 1 < |z| < 2$.

12.8 REFERENCE BOOKS:

1) M.R. Spiegel 'Complex variables', McGraw-Hill Book Co., 1964.
2) E. Kreyszig 'Advanced engineering mathematics', Wiley Eastern Pvt., Ltd., 1971.
3) B.D. Gupta 'Mathematical Physics', Vikas publishing House, Sahibabad, 1980.

Prof. G. Naga Raju

LESSON-13

CALCULUS OF RESIDUES

13.0 AIM AND OBJECTIVE:

The primary aim of this lesson is to understand the concepts of Calculus of Residues, Cauchy's Residue theorem, Evaluation of Residues, Evaluation of contour integrals. After completing this chapter, students should be able to employ various techniques for evaluating residues, including formulas and limit methods; and enhance their analytical and problem-solving skills through the application of the Calculus of Residues.

STRUCTURE:

13.1 INTRODUCTION-CALCULUS OF RESIDUES

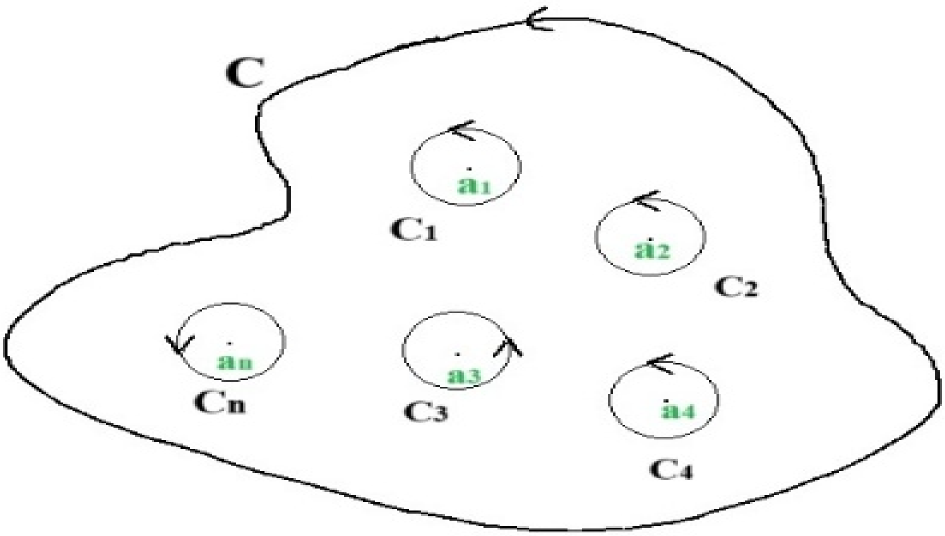
The Calculus of Residues is a branch of mathematics that provides us with a profound understanding of the behavior of complex functions, particularly around their singularities, and offers a remarkably efficient way to evaluate complex integrals. Applications of these topics include electromagnetism, quantum mechanics, and fluid dynamics.

13.2 CAUCHY'S RESIDUE THEOREM:

Statement: Let a function f(z) be analytic inside a closed C except at a finite no. of poles within C then

$$\int_C f(z)dz = 2\pi i\ (sum\ of\ the\ residues\ at\ the\ poles\ within\ C)$$

Proof: Let f(z) is analytic in a closed curve C except at finite number of poles within C, say a_1, a_2, …. a_n i.e., f(z) is analytic within and on thcicle C except at a_1, a_2, …. a_n. Now draw small non-interacting circles with centres a_1, a_2, …. a_n respectively and radii is so small that lie entirely within the closed circle C then f(z) is analytic in the bounded region bounded by C and C_1, C_2, C_3, … C_n.



Applying Cauchy's theorem,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

$$= 2\pi i\, Res\ of\ f(at\ z = a_1) + 2\pi i\, Res\ of\ f(at\ z = a_2) + \cdots + 2\pi i\, Res\ of\ f(at\ z = a_n)$$

$$= 2\pi i\, [Res\ of\ f(a_1) + Res\ of\ f(a_2) + \cdots + Res\ of\ f(a_n)$$

$$\int_C f(z)dz = 2\pi i\ (sum\ of\ the\ residues\ at\ the\ poles\ within\ C)$$

13.2 EVALUATION OF RESIDUE:

1) **f(z) has a simple pole at z = a.** In this case the Laurent's expansion of f(z) is

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n + \frac{d_1}{z-a}.$$

Therefore,

$$\boxed{\underset{z=a}{res}\ f(z) = d_1 = \lim_{z\to a} (z-a)\ f(z)}$$

If $f(z) = \frac{g(z)}{h(z)}$, where h(z) has a simple zero at z = a and g(a) ≠ 0, then f(z) has a simple pole at z = a. In this case,

$$d_1 = \lim_{z\to a}(z-a)\frac{g(z)}{h(z)} = \lim_{z\to a}\frac{(z-a)\,g(z)}{h(z)-h(a)} = \frac{g(a)}{h'(a)}$$

> *If* $f(z) = \dfrac{g(z)}{h(z)}$, *where* $h(z)$ *has a simple zero at* $z = a$, $g(a) \neq 0$, *then*
>
> $$d_1 = \frac{g(a)}{h'(a)}$$

2) **f(z) has a pole of order m at z = a.**

Denote$\varphi(z) = (z-a)^m f(z)$. Then φ(z) is analytic in some neighbourhood of the point a. Let, the Taylor series expansion of φ(z) in this neighbourhood be

$$\varphi(z) = \varphi(a) + \varphi'^{(a)}(z-a) + \cdots + \frac{\varphi^{(m-1)}(a)}{(m-a)!}(z-a)^{m-1} + \cdots$$

Therefore, the Laurent's expansion of f(z) in the deleted neighbourhood of a is

$f(z) = \frac{\varphi(z)}{(z-a)^m} = \frac{\varphi^{(m-1)}(a)}{(m-a)!}\frac{1}{(z-a)} + \cdots + \varphi(z)\frac{1}{(z-a)^m}$ + non-negative powers of (z-a)

$$d_1 = coefficient\ of\ \frac{1}{(z-a)^m} = \frac{\varphi^{(m-1)}(a)}{(m-a)!} = \frac{1}{(m-a)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]_{z=a}$$

13.3 EVALUATION OF CONTOUR INTEGRALS:

Type 1: Integrals of the form$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

If we take z = $e^{i\theta}$, then cosθ = $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$.

Substituting for sinθ, cosθ and dθ the definite integral transforms into the following contour integral

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta = \int_{|z|=1} f(z)dz$$

where $f(z) = \frac{1}{iz}\left[F\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\right]$

Apply Residue theorem to evaluate

$$\int_{|z|=1} f(z)dz$$

13.4 PROBLEM:

1) Consider

$$\int_0^{2\pi} \frac{1}{1+3(\cos t)^2} dt.$$

$$\int_0^{2\pi} \frac{1}{1+3(\cos t)^2} dt = \int_{|z|=1} \frac{1}{1+3\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2} \frac{dz}{iz}$$

$$= -4i\int_{|z|=1} \frac{z}{3z^4+10z^2+3} dz$$

$$= -4i\int_{|z|=1} \frac{z}{3(z+\sqrt{3}i)(z-\sqrt{3}i)\left(z+\frac{i}{\sqrt{3}}\right)\left(z-\frac{i}{\sqrt{3}}\right)} dz$$

$$= -\frac{4}{3}i\int_{|z|=1} \frac{z}{3(z+\sqrt{3}i)(z-\sqrt{3}i)\left(z+\frac{i}{\sqrt{3}}\right)\left(z-\frac{i}{\sqrt{3}}\right)} dz$$

$$-\frac{4}{3}i \times 2\pi i\left\{Res\left(f, \frac{i}{\sqrt{3}}\right) + Res\left(f, -\frac{i}{\sqrt{3}}\right)\right\}$$

2) f(z) = cot z. (Problem related to Evaluation of residues)

The function $f(z) = \cot z = \frac{\cos z}{\sin z}$ has simple poles at the points z = nπ & cos nπ≠ 0. Therefore,

$$\underset{z=n\pi}{res}\ f(z) = \frac{\cos n\pi}{\cos n\pi} = 1.$$

(Note the difficulty in finding Laurent's expansion of cot z in deleted neighbourhood of the points nπ.

13.5 SUMMARY:

This lesson introduces the Calculus of Residues, a powerful tool for evaluating complex integrals. It begins with the concept of residues, quantifying the behavior of functions near isolated singularities, and demonstrates methods for their evaluation. The

Cauchy Residue Theorem then connects these residues to contour integrals, stating that the enclosed residues determine the integral of a meromorphic function around a closed curve. This theorem enables the efficient evaluation of contour integrals, which often simplifies to calculating residues, especially for real integrals transformed into the complex plane. The lesson emphasizes the application of residue calculus to solve integrals that are otherwise difficult or impossible to evaluate using traditional real calculus methods.

13.6 TECHNICAL TERMS:

Calculus of Residues-Cauchy's Residue Theorem-Evaluation of Residues-Evaluation of Contour Integrals

13.7 SELF-ASSESSMENT QUESTIONS:

1) Evaluate real integral $\int_{-\infty}^{\infty} \frac{1}{x^2+4x+5} dx$.

2) Evaluate real integral $\int_{0}^{2\pi} \frac{1}{2+\cos(\theta)} dx$.

13.8 SUGGESTED BOOKS:

1) M.R. Spiegel 'Complex variables', McGraw-Hill Book co., 1964.

2) E. Kreyszig 'Advanced Engineering Mathematics', Wiley Eastern Pvt., Ltd., 1971.

3) B.D. Gupta 'Mathematical Physics', Vikas Publishing House, Sahibabad, 1980.

Prof. M. Rami Reddy

LESSON-14

TENSOR ANALYSIS

14.0 AIM AND OBJECTIVE:

The aim of this lesson is to equip students with a foundational understanding of tensor classification by distinguishing between contravariant, covariant, and mixed tensors based on their transformation properties under coordinate changes, enabling them to correctly identify and manipulate these tensors in subsequent tensor analysis applications and physical contexts.

STRUCTURE:

14.1 INTRODUCTION OF TENSOR:

Scalars are specified by magnitude only, Vectors have magnitude as well as direction. But Tensors are associated with magnitude and two or more directions.

Tensor Analysis is suitable for Mathematical formulation of Natural Laws in forms which are invariant with respect to different frames of reference. That is why Einstein used Tensors for the formulation of his Theory of Relativity.

A scalar is a zero-order tensor. A vector is a first-order tensor. A matrix is a second order tensor. For example, consider the stress tensorτ.

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Another way to write a vector is in Cartesian form:

$\vec{X} = X\hat{\imath} + X\hat{\jmath} + X\hat{k} = (x, y, z)$ .... (1)

The coordinates x, y and z can also be written as x1, x2, x3. Thus the vector can be written as

$\vec{X} = (X_1, X_2, X_3)$ .... (2)

Or as

$\vec{X} = (X_i), i = 1,2,3$ .... (3)

or in index notation, simply as

$\vec{X} = X_i$ .... (4)

Where i is understood to be a dummy variable running from 1 to 3.

Thus xi, xj and xp all refer to the same vector (x_1, x_2 and x_3), as the index (subscript) always runs from 1 to 3.

14.2 CONTRAVARIANT TENSORS:

A contravariant tensor is a tensor having specific transformation properties (cf., a covariant tensor). To examine the transformation properties of a contravariant tensor, first consider a tensor of rank 1 (a vector).

Dr = $dx_1\hat{x}_1 + dx_2\hat{x}_2 + dx_3\hat{x}_3$ ....(5)

for which

$dx'_i = \frac{dx'_i}{dx_j} dx_j$ .... (6)

Now let $A_i = dx_i$, then any set of quantities which transform according to

$A'_i = \frac{dx'_i}{dx_j} A_j$ .... (7)

or, defining

$a_{ij} = \frac{dx'_i}{dx_j}$ .... (8)

according to

$A'_i = a_{ij} A_j$ ……………… (9)

is a contravariant tensor. Contravariant tensors are indicated with raised indices, i.e., a^{μ}.

Contravariant Tensors of Second Rank:

Let us consider $(n)^2$ quantities A^{ij} (here i and j take the values from 1 to n independently) in a system of variables x^i and let these quantities have values $A'^{\mu\nu}$ in another system of variables x'^{μ}.

If these quantities obey the transformation equations

$A'^{\mu\nu} = (\partial x'^{\mu} / \partial x^i)(\partial x'^{\nu} / \partial x^j) A^{ij}$ ………………… (5)

then the quantities A^{ij} are said to be the components of a contravariant tensor of second rank.

14.3 COVARIANT TENSORS:

Covariant tensors are a type of tensor with differing transformation properties, denoted a_y. However, in three-dimensional Euclidean space,

$$\frac{\partial x_j}{\partial x'_j} = \frac{\partial x'_i}{\partial x_i} \equiv a_{ij} \qquad \text{…………………..} (10)$$

for i, j = 1, 2, 3, meaning that contravariant and covariant tensors are equivalent. Such tensors are known as Cartesian tensor. The two types of tensors do differ in higher dimensions, however.

Contravariant four-vectors satisfy

$$a^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu} \qquad \text{…………………} (11)$$

where Λ is a Lorentz tensor.

To turn a covariant tensor a_y into a contravariant tensor a^{μ} (index raising), use the metric tensor $g^{\mu\nu}$ to write

$$g^{\mu\nu} a_{\nu} = a^{\mu} \qquad \text{…………….} (12)$$

Covariant and contravariant indices can be used simultaneously in a mixed tensor.

14.4 MIXED TENSOR:

A tensor having contravariant and covariant indices. In tensor analysis, a mixed tensor is a tensor which is neither strictly covariant nor strictly contravariant; at least one of the indices of a mixed tensor will be a subscript (covariant) and at least one of the indices will be a superscript (contravariant).

A mixed tensor of type or valence $\binom{M}{N}$, also written "type (M, N)", with both $M > 0$ and $N > 0$, is a tensor which has M contravariant indices and N covariant indices. Such a tensor can be defined as a linear function which maps an $(M + N)$ tuple of M one-forms and N vectors to a scalar.

Covariant Tensors of Second Rank:

Let us consider $(n)^2$ quantities A^{ij} (here i and j take the values from 1 to n independently) in a system of variables x^i and let these quantities have values $A'_{\mu v}$ in another system of variables x'^{μ}.

If these quantities obey the transformation equations

$A'_{\mu v} = (\partial x^i/\partial x'^{\mu}) (\partial x^j /\partial x'^{v}) A_{ij}$ ………………………………………… (6)

then the quantities A_{ij} are said to be the components of a covariant tensor of second rank.

14.5 PROBLEMS:

1. Show that the law of transformation for a contravariant vector is transitive.

We have $A'^{\mu} = \frac{\partial x'_{\mu}}{\partial x_{\alpha}} A^{\alpha}$

Let $A''^{\mu} = \frac{\partial x''_{\mu}}{\partial x'_{\alpha}} A'^{\alpha}$

$\therefore A''^{\mu} = \frac{\partial x''_{\mu}}{\partial x'_{\beta}} A^{\beta} = \frac{\partial x''_{\mu}}{\partial x'_{\beta}} \frac{\partial x'_{\beta}}{\partial x_{\alpha}} A^{\alpha} = \frac{\partial x''_{\mu}}{\partial x_{\alpha}} A^{\alpha}$

which shows that contravariant law is transitive.

14.6 SUMMARY

This lesson, Tensor Analysis focuses on classifying tensors based on their transformation properties under coordinate changes. Contravariant tensors, denoted with upper indices, transform "against" the coordinate change, while covariant tensors, denoted with lower indices, transform "with" the coordinate change. Mixed tensors, possessing both upper and lower indices, exhibit a combination of these transformations. Understanding these distinctions is crucial for expressing physical laws and geometric relationships in a coordinate-independent manner, allowing for consistent descriptions across different reference frames.

14.7 TECHNICAL TERMS:

Tensors-Contravariant Tensors-Covariant Tensors-Mixed Tensors.

14.8 SELF-ASSESSMENT QUESTIONS:

1) A contravariant vector Ai in a 3-dimensional Cartesian coordinate system (x^1, x^2, x^3) has components: $A^1 = x^1 + 2x^2$, $A^2 = x^3$, $A^3 = (x^1)^2 - x^3$. The coordinate system is transformed to a new system $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ according to the following transformation: $\bar{x}^1 = x^1 - x^2 + x^3$, $\bar{x}^2 = 2x^2 - x^3$, $\bar{x}^3 = x^1 + x^3$. Determine the components of the transformed contravariant vector $\bar{A}^i$ in the new coordinate system. Express your answer in terms of the original coordinates (x^1, x^2, x^3).

2) A mixed tensor T^i_j in a 2-dimensional coordinate system (x^1,x^2) has the following components: $T^1_1 = x^1x^2$, $T^1_2 = (x^2)^2$, $T^2_1 = -x^1$, $T^2_2 = x^1 + x^2$. The coordinate system is transformed to a new system $(\bar{x}^1, \bar{x}^2)$ according to the following transformation: $\bar{x}^1 = 2x^1 + x^2$, $\bar{x}^2 = x^1 - x^2$, Calculate the components of the transformed mixed tensor $\bar{T}^i_j$ in the new coordinate system. Express your answer in terms of the new coordinates $(\bar{x}^1, \bar{x}^2)$.

14.9 SUGGESTED BOOKS:

1) M.R. Spiegel 'Complex variables', McGraw-Hill Book co., 1964.
2) E. Kreyszig 'Advanced engineering mathematics', Wiley Eastern Pvt., Ltd., 1971.
3) B.D. Gupta 'Mathematical Physics', Vikas Publishing House, Sahibabad, 1980.

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LESSON-15

TENSOR ANALYSIS-II

15.0 AIM AND OBJECTIVE:

The aim of this lesson is to establish a clear understanding of tensor classification by exploring the concept of tensor rank, the distinction between symmetric and anti-symmetric tensors, and the importance of invariant tensors, enabling students to analyze and solve problems involving tensors while enhancing their analytical skills in various scientific and mathematical contexts.

STRUCTURE:

15.1 INTRODUCTION:

This lesson introduces fundamental concepts related to the structure and properties of tensors, crucial for advanced applications in physics and mathematics. We begin by defining the **rank of a tensor**, which signifies the number of indices and dictates its transformation behavior. We then explore two special classes of tensors: **symmetric tensors**, which remain unchanged under the permutation of their indices, and **anti-symmetric (or skew-symmetric) tensors**, which change sign upon index permutation. Finally, we discuss **invariant tensors**, tensors whose components remain the same across all coordinate transformations, highlighting their importance in expressing fundamental physical laws and geometric properties in a coordinate-independent manner. This lesson builds a foundational understanding of tensor classification and properties, essential for further study in fields like general relativity, continuum mechanics, and differential geometry.

15.2 RANK OF TENSOR:

Total number of contravariant and covariant indices of a tensor. The rank R of a tensor is independent of the number of dimensions N of the underlying space. An intuitive way to think of the rank of a tensor is as follows: First, consider intuitively that a tensor represents a physical entity which may be characterized by magnitude and multiple directions simultaneously. Therefore, the number of simultaneous directions is denoted R and is called the rank of the tensor in question. In N-dimensional space, it follows that a rank-0 tensor (i.e., a scalar) can be represented by $N^0 = 1$ number since scalars represent quantities with magnitude and no direction; similarly, a rank-1 tensor (i.e., a vector) in N-dimensional space can be represented by $N^1 = N$ numbers and a general tensor by N^R numbers. From this perspective, a rank-2 tensor (one that requires N^2 numbers to describe) is equivalent, mathematically, to an N X N matrix.

Rank	Object
0	scalar
1	vector
2	N X N matrix
≥3	tensor

The above table gives the most common nomenclature associated to tensors of various rank. Some care must be exhibited, however, because the above nomenclature is hardly uniform across the literature. For example, some authors refer to tensors of rank 2 as dyads, a term used completely independently of the related term dyadic used to describe vector direct products. Following such convention, authors also use the terms triad, tetrad, etc., to refer to tensors of rank 3, rank 4, etc.

15.2 SYMMETRIC AND ANTI-SYMMETRIC TENSORS:

15.2.1 (a) Symmetric Tensors:

If two contravariant or covariant indices can be interchanged without altering the tensor, then the tensor is said to be symmetric with respect to these two indices.

For example if $A^{ij} = A^{ji}$ or $A_{ij} = A_{ji}$ .... (13)

then the contravariant tensor of second rank A^{ij} or covariant tensor A_{ij} is said to be symmetric

For a tensor A_l^{ijk} of higher rank

If $A_l^{ijk} = A_l^{jik}$

Then the tensor A_l^{ijk} is said to be symmetric with respect to indices i and j.

15.2.1.1 Theorem 1:

The symmetry property of a tensor in independent of co-ordinate system used.

If tensor A_l^{ijk} is symmetric with respect to first indices i and j, we have

$$A_l^{ijk} = A_l^{jik} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(14)}$$

$$\text{Now } A'^{\mu\nu\sigma}_{p} = (\partial x'^{\mu} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{ijk}$$

$$= (\partial x'^{\mu} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{jik}$$

using eq (14), Again interchanging the dummy indices i and j, we get

$$A'^{\mu\nu\sigma}_{p} = (\partial x'^{\mu} / \partial x^{j})(\partial x'^{\nu} / \partial x^{i})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{ijk}$$

$$= (\partial x'^{\mu} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{ijk}$$

$$= A'^{\nu\mu\sigma}_{p}$$

i.e. given tensor is gain symmetric with respect to first two indices in new co-ordinate system. Thus the symmetry property of a tensor is independent of coordinate system.

15.2.1.2 Theorem 2:

Symmetry is not preserved with respect to two indices, one contravariant and the other covariant.

Let A_l^{i} be symmetric with respect to two indices, one contravariant i and the other covariant l, then we have

$$A_l^{ijk} = A_i^{ljk} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(15)}$$

$$A'^{\mu\nu\sigma}_{p} = (\partial x'^{\mu} / \partial x^{l})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{ijk}$$

$$= (\partial x'^{\mu} / \partial x^{l})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_i^{ljk}$$

Using eq. (15), Again interchanging the dummy indices i and l, we get

$A^{l}{}_{p}{}^{\mu\nu\sigma} = (\partial x'^{\mu} / \partial x^{l})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{i} / \partial x'^{p})\ A_{i}{}^{ljk}$

$A^{l}{}_{\mu}{}^{\mu\nu\sigma} = (\partial x'^{p} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{\mu})\ A_{l}{}^{ijk}$

Thus, $A^{l}{}_{p}{}^{\mu\nu\sigma} \neq A^{l}{}_{\mu}{}^{\mu\nu\sigma}$

15.2.2. Antisymmetric Tensors or Skew-Symmetric Tensors.

A tensor, whose each component alters in sign but not in magnitude when two contravariant or covariant indices are interchanged, is said to be skew symmetric or anti-symmetric with respect to these two indices.

For example if $A^{ij} = -A$ (or) $A_{ij} = -A_{ji}$ ……………………………… (16)

Then contravariant tensor A^{ij} or covariant tensor A_{ij} of second rank is anti-symmetric or for a tensor of higher rank $A_{l}{}^{ijk}$

If $A_{l}{}^{ijk} = -A_{l}{}^{ikj}$ then tensor $A_{l}{}^{ijk}$ is antisymmetric with respect to indices j and k.

The skew-symmetry property of a tensor is also independent of the choice of coordinate system. So if a tensor is skew symmetric with respect to two indices in any coordinate system, it remains skew-symmetric with respect to these two indices in any other coordinate system.

If all the indices of a contravariant or covariant tensor can be interchanged so that its components

change sign at each interchange of a pair of indices, the tensor is said to be anti-symmetric,

i.e., $A^{ijk} = -A^{jik} = +A^{jki}$

Thus we may state that a contravariant or covariant tensor is anti-symmetric if its components change sign under an odd permutation of its indices and do not change sign under an even permutation of its indices.

15.3 INVARIANT TENSORS:

Invariants of a tensor are scalar functions of the tensor components which remain constant under a basis change. That is to say, the invariant has the same value when computed in two arbitrary bases $\{e_1,e_2,e_3\}$ and $\{m_1,m_2,m_3\}$. A symmetric second order tensor always has three independent invariants.

(OR)

Invariant tensors are tensors that remain unchanged under a specific group of transformations. These transformations could be rotations, Lorentz transformations, or other symmetry operations depending on the physical or mathematical context.

Definition: A tensor $T^{\mu\nu\cdots}$ is said to be invariant under a transformation Λ if it satisfies:

$$T'^{\mu\nu\cdots} = \Lambda_{\rho}{}^{\mu}\,\Lambda_{\sigma}{}^{\nu}\cdots\, T^{\rho\sigma\cdots} = T^{\mu\nu\cdots}$$

for all transformations in a given symmetry group.

Examples of Invariant Tensors:

- **Kronecker Delta ($\delta^{\mu\nu}$)**: The identity tensor in any metric space, which remains nchanged under any orthogonal transformation.
- **Levi-Civita Symbol ($\epsilon^{\mu\nu\lambda\cdots}$)**: An antisymmetric tensor that remains invariant up to a sign under proper transformations (like special orthogonal groups).
- **Metric Tensor ($g_{\mu\nu}$)**: Invariant under coordinate transformations that preserve the metricstructure (e.g., Lorentz transformations in relativity).

15.5 SUMMARY:

This lesson categorizes tensors by their rank (number of indices), explores the special properties of symmetric and anti-symmetric tensors based on index permutation behavior, and highlights the significance of invariant tensors, which remain unchanged under coordinate transformations, providing a foundation for understanding tensor properties and their applications across diverse fields.

15.6 TECHNICAL TERMS:

Rank of a tensor - Symmetric and anti-symmetric tensors - Invariant tensors

15.7 SELF-ASSESSMENT QUESTIONS:

1) Define the rank of a tensor and provide examples of tensors with ranks 0, 1, and 2.
2) Explain the difference between symmetric and anti-symmetric tensors, and give a mathematical expression for each.

15.8 SUGGESTED BOOKS:

1) M.R. Spiegel ‘Complex variables’, McGraw-Hill Book co., 1964.

2) E. Kreyszig ‘Advanced engineering mathematics’, Wiley Eastern Pvt., Ltd., 1971.

3) B.D. Gupta ‘Mathematical Physics’, Vikas Publishing House, Sahibabad, 1980.

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LESSON-16

TENSOR ANALYSIS–III

16.0 AIM AND OBJECTIVE:

The aim of this lesson is to provide students with the operational tools necessary to manipulate tensors, specifically through addition, multiplication (outer and inner products), contraction, and the quotient law, enabling them to combine and analyze tensor quantities effectively and to determine tensor character from relationships between quantities.

STRUCTURE:

16.1 INTRODUCTION:

This lesson introduces the fundamental algebraic operations applied to tensors, enabling their manipulation and combination. We will explore tensor addition, which requires tensors of the same type, and multiplication, encompassing both outer products, which increase tensor rank, and inner products, which reduce it. The concepts of contraction, a process that reduces tensor rank by summing over repeated indices, and the quotient law, which helps determine if a quantity is a tensor based on its interaction with known tensors, are also covered, providing essential tools for tensor calculus and its applications in various scientific fields.

16.2 ADDITION AND MULTIPLICATION OF TENSORS :

In tensor algebra, addition and multiplication operate similarly to matrices but with more dimensions. Let's break them down:

1. Addition of Tensors:

- **Rule:** Two tensors can be added if they have the **same shape**.
- **Element-wise Operation:** Addition happens component-wise.

Let's take two rank-2 tensors (matrices):

$$A = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}, \qquad A = \begin{matrix} 5 & 6 \\ 7 & 8 \end{matrix}$$

Addition:

$$A + B = \begin{matrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{matrix} = \begin{matrix} 6 & 8 \\ 10 & 12 \end{matrix}$$

For higher-rank tensors, the same rule applies: element-wise addition as long as the shapes match.

2. Multiplication of Tensors:

There are multiple ways to multiply tensors, depending on the context:

(a) Element-wise (Hadamard) Multiplication

- **Rule:** Tensors must have the **same shape**.
- **Operation:** Multiply corresponding elements.

Examples:

$$A \odot B = \begin{matrix} 1 \cdot 5 & 2 \cdot 6 \\ 3 \cdot 7 & 4 \cdot 8 \end{matrix} = \begin{matrix} 5 & 12 \\ 21 & 32 \end{matrix}$$

(b) Tensor (Outer) Product

- **Rule:** No shape restriction.
- **Operation:** Each element of the first tensor multiplies the entire second tensor.

Example:

If A is a 1D tensor (vector) and B is also a 1D tensor:

A=[1,2],B=[3,4]

The outer product:

$$A \otimes B = \begin{matrix} 1 \cdot 3 & 1 \cdot 4 \\ 2 \cdot 3 & 2 \cdot 4 \end{matrix} = \begin{matrix} 3 & 4 \\ 6 & 8 \end{matrix}$$

Outer and Inner Products:

Both **outer** and **inner** products are fundamental tensor operations used in linear algebra and machine learning. Let's break them down with definitions and examples.

16.3 OUTER AND INNER PRODUCT:

1. Outer Product:

The outer product of two tensors creates a higher-dimensional tensor. It is defined as:

$$(A\otimes B)_{i,j} = A_i B_j$$

Key Properties:

- Dimension increases: If AAA is of shape (m,)(m,)(m,) and BBB is of shape (n,)(n,)(n,), the result is a tensor of shape (m,n)(m, n)(m,n).
- No summation occurs: Every element of AAA multiplies every element of BBB independently.

Example (Vectors):

$$A = \begin{bmatrix}1\\2\\3\end{bmatrix}, B = \begin{bmatrix}4\\5\end{bmatrix}$$

$$A\otimes B = \begin{bmatrix}1\,X\,4 & 1\,X\,5\\2\,X\,4 & 2\,X\,5\\3\,X\,4 & 3\,X\,5\end{bmatrix} = \begin{bmatrix}4 & 5\\8 & 10\\12 & 15\end{bmatrix}$$

For higher-order tensors, the same rule applies: each element of AAA multiplies every element of BBB, creating a larger-dimensional tensor.

2. Inner Product:

In tensor analysis, the inner product is a fundamental operation that combines two tensors to form a scalar.

Give two tensors A and B, the inner product is denoted by

$$A.B$$

The inner product is a scalar value that results from contracting the two tensors along a shared index.

The inner product of two tensors A and B can be mathematically defined as

A.B=A^iB_i

Where

A^i is the contravariant component of tensor A

B_ is the covariant component of tensor B

The summation convention is implied, meaning that the index i is summed over.

16.4 CONTRACTION OF TENSORS:

Given a tensor $T^{\mu\nu}$, the contraction over the indices μ and ν would be written as:

$$T=T_{\mu}^{\mu} = \Sigma_{\mu} T^{\mu\nu}$$

This operation involves summing the components of the tensor over the repeated index, leading to a scalar quantity (if it is fully contracted).

In a more general setting, contraction can also refer to the summation of indices in a product of tensors. For example, in the contraction of a product of two tensors $A^{\mu\nu}$ and $B_{\mu\nu}$ one sums over the indices μ and ν:

$$C = A^{\mu\nu} B_{\mu\nu} = \Sigma_{\mu\nu} A^{\mu\nu} B_{\mu\nu}$$

This results in a scalar quantity if both indices are contracted.

16.5 QUOTIENT LAW:

In tensor analysis, the **quotient law** describes the way tensor operations behave when applied to a quotient of two tensors. It can be used in the context of covariant and contravariant tensors or the division of tensors by scalar quantities.

If T is a tensor and s is a scalar, then the quotient rule expresses the behavior of the tensor T divided by the scalar s:

$$\frac{T}{s} \text{ implies } \frac{T^{\mu\nu}}{s} = \frac{T^{\mu\nu}}{s}$$

This is essentially the operation of dividing each component of the tensor by the scalar. For more complicated expressions involving tensor products, the quotient rule can extend to ensure the correct handling of indices.

Practical Example (in General Relativity):

Consider the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ where:

$$g_{\mu\nu} g^{\nu\alpha} = \delta_{\mu}^{\alpha}$$

In this case, contraction can be used to sum over indices in equations like the Ricci tensor $R_{\mu\nu}$ or to contract components of the Einstein field equations in general relativity.

16.6 SUMMARY:

This lesson covers the essential algebraic operations for tensors: addition (requiring like tensors), outer multiplication (increasing rank), inner multiplication (reducing rank), contraction (summing over indices to lower rank), and the quotient law (determining tensor nature from interactions), providing the tools to combine, manipulate, and analyze tensors effectively in various scientific applications.

16.7 TECHNICAL TERMS:

Addition and multiplication of tensors - Outer and inner products - Contraction of tensors - quotient law.

16.8 SELF-ASSESSMENT QUESTIONS

1) Given a contravariant tensor A^{ij} and a covariant tensor B_k in a 3-dimensional coordinate system, where: $A^{ij} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$ and $B_k = \begin{pmatrix} k \\ k^2 \\ -k \end{pmatrix}$
   a) Calculate the outer product $C^{ijk} = A^{ij}B_k$.
   b) Calculate the contraction of the tensor $D^i_j = A^{ij}B_k$ with respect to the indices i and j.
2) In a 2-dimensional coordinate system, let T_{ij} be a known covariant tensor. It is given that for any arbitrary contravariant vector V^j, the quantity $W_i = T_{ij}V^j$ is a covariant vector.
   a) State the quotient law and explain its significance in determining tensor character.
   b) Using the given information and the quotient law, prove that T_{ij} is indeed a covariant tensor.

16.9 SUGGESTED BOOKS:

1) M.R. Spiegel 'Complex variables', McGraw-Hill Book Co., 1964.
2) E. Kreyszig 'Advanced Engineering Mathematics', Wiley Eastern Pvt., Ltd., 1971.
3) B.D. Gupta 'Mathematical Physics', Vikas Publishing House, Sahibabad, 1980.

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